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MISCELLANEOUS

★**Cartwright, Mary L. The mathematical mind.** Oxford University Press, London-New York-Toronto, 1955. 28 pp. 2s.6d.
Lecture delivered in May 1955.

★**Famous Problems and other monographs. Famous problems of elementary geometry, by F. Klein. From determinant to tensor, by W. F. Sheppard. Introduction to combinatory analysis, by P. A. MacMahon. Three lectures on Fermat's last theorem, by L. J. Mordell.** Chelsea Publishing Co., New York, 1955. i+338 pp. $3.25.
Reprint by photo-offset of the four books of the subtitle above [2nd ed., Stechert, New York, 1930; Oxford, 1923; Cambridge, 1920; Cambridge, 1921].

★**Enzyklopädie der Elementarmathematik. Bd. 1. Arithmetik.** Redaktion: P. S. Alexandroff, A. I. Markuschewitsch, und A. J. Chintschin. Deutscher Verlag der Wissenschaften, Berlin, 1954. xi+403 pp.
Translation of Enciklopediya èlementarnoĭ matematiki, Kniga 1, Arifmetika [Gostehizdat, Moscow, 1951]. This volume contains the following expository articles: The origin of the systems of notation for numbers, by G. Bašmakova and A. P. Yuškevič; Sets, groups, rings, fields, by I. V. Proskuryakov; The elements of number theory, by A. I. Hinčin; and Mental and written computation, by V. M. Bradis.

★**Hasse, Helmut. Proben mathematischer Forschung in allgemeinverständlicher Behandlung.** Otto Salle Verlag, Frankfurt am Main-Pinneberg, 1955. viii+103 pp. DM 6.80.
The author contends that a mathematician doing research is comparable to a musician composing music, and that the intelligent audience for the former's output is not necessarily restricted to those making a profession of mathematics. He reproaches the traditional methods of instruction in mathematics with not developing the inherent disposition to the enjoyment of mathematics ("mathematicality") in those who possess it. Although he grants that mathematics has its "late Beethoven quartets" which are only for the initiated, he also believes that it has its "Schubert Lieder" which are immediately accessible to those possessing mathematicality. This book is the author's attempt to provide such Lieder. He has aimed at selecting topics which are simple, central to mathematics (in particular, not entertainment), can be treated in a self-contained manner, and span a number of fields of mathematics.
The first topic treated is prime numbers: e.g., the sieve of Eratosthenes, infinitude of primes. Next comes a section on various geometrical extremum problems (the importance of establishing existence is pointed up by the well-known false proof that 1 is the largest integer). The third section is concerned with a variety of topics on incommensurability, irrationality, and transcendence. The last section is devoted to map-coloring problems and Euler's polyhedron theorem. The five-color theorem is proved. Coloring maps on surfaces of higher genus, theorems equivalent to the four-color theorem, and an application in number theory are also discussed.
There seems little question but that the author has written a book which will give pleasure to many readers. Whether the individual topics are really as readily accessible as a Schubert Lieder may be questioned. The momentary lapses of attention which do not necessarily prevent the enjoyment of music might not be tolerable here; but perhaps this is part of the essence of mathematicality. *J. V. Wehausen* (Providence, R.I.).

Gericke, H. Die Stellung der Mathematik in der Kulturgeschichte. Nachr. Giessener Hochschulges. 23 (1954), 116–126.

van Dantzig, D. The function of mathematics in modern society and its consequence for the teaching of mathematics. Euclides, Groningen 31 (1955/56), 88–102.

HISTORY, BIOGRAPHY

★**Gericke, H. Zur Geschichte der Mathematik an der Universität Freiburg i. Br.** Verlag Eberhard Albert, Freiburg im Breisgau, 1955. 88 pp. DM 4.50.

★**Markuschewitsch, A. I. Skizzen zur Geschichte der analytischen Funktionen.** Deutscher Verlag der Wissenschaften, Berlin, 1955. viii+139 pp. DM 8.70.
Translation of Markuševič's "Očerki po istorii teorii analitičeskih funkciĭ [Gostehizdat, Moscow-Leningrad, 1951; MR 14, 2].

Pihl, Mogens. From the history of the lever principle. Nordisk Mat. Tidskr. 3 (1955), 148–156, 183. (Danish. English summary)

★**Polvani, Giovanni. Il moto della terra, filo storico della relatività.** Cinquant'anni di Relatività, 1905–1955, pp. 3–28. Editrice Universitaria, Firenze, 1955.

★**Straneo, Paolo. Genesi ed evoluzione della concezione relativistica di Albert Einstein.** Cinquant'anni di Relatività, 1905–1955, pp. 29–134. Editrice Universitaria, Firenze, 1955.

Bilimovitch, Anton. Apollonius theorem on station of the planet. Glas Srpske Akad. Nauka 206, Od. Prirod.-Mat. Nauka (N.S.) 5 (1953), 49–56. (Serbo-Croatian. English summary)

Milne, William P. Obituary: Selig Brodetsky. J. London Math. Soc. **31** (1956), 121–125.

★ **Carathéodory, Constantin. Gesammelte mathematische Schriften. Bd. 3.** Herausgegeben im Auftrag und mit Unterstützung der Bayerischen Akademie der Wissenschaften. C. H. Beck'sche Verlagsbuchhandlung, München, 1955. ix+464 pp. DM 46.00.

This volume, edited by H. Tietze, consists of papers on functions of a complex variable and allied topics by Carathéodory. Further papers by Carathéodory in this field will be contained in the fourth volume of his collected works. The present volume contains the papers of Carathéodory on the following topics: the Picard theorem; the coefficient problem for positive harmonic functions; the Schwarz lemma; conformal mapping, existence theorems (the paper on the Dirichlet problem [Amer. J. Math. **59** (1937), 709–731] is included in this section); conformal mapping of variable domains; mappings defined by systems of analytic functions of several variables; the "Starrheitssatz" (written jointly with G. Aumann); mappings which preserve circles. *M. Heins.*

Speziali, Pierre. Une correspondance inédite entre Clairaut et Cramer. Rev. Hist. Sci. Appl. **8** (1955), 193–237.

Ferrar, W. L. Obituary: Arthur Lee Dixon. J. London Math. Soc. **31** (1956), 126–128.

★ **Einstein, Albert. Le memorie fondamentali di Albert Einstein.** Cinquant'anni di Relatività, 1905–1955, pp. 477–611. Editrice Universitaria, Firenze, 1955.

Translations into Italian of the following works of Einstein: Ann. Physik (4) **17** (1905), 891–921; **18** (1905), 639–641; **49** (1916), 769–822; S.B. Preuss. Akad. Wiss. **1916**, 1111–1116; **1917**, 142–152; The meaning of relativity, 4th ed., Appendix II, Supplement to Appendix II [Princeton, 1953; MR **14**, 805; **15**, 357].

Dugas, René. Einstein et Gibbs devant la thermodynamique statistique. C. R. Acad. Sci. Paris **241** (1955), 1685–1687.

Obláth, Richárd. Commémoration de Gauss (1777-1855). Mat. Lapok **6** (1955), 221–240. (Hungarian. French summary)

Turán, Pál. The life and mathematical work of Géza Grünwald. Mat. Lapok **6** (1955), 6–26. (Hungarian. Russian and English summaries)

A list of Grünwald's mathematical papers is included.

Hinrichsen, J. J. L., Thielman, H. P., and Gouwens, Cornelius. Dio Lewis Holl, 1895-1954. Proc. Iowa Acad. Sci. **62** (1955), 71–72.

Péter, Rózsa. Die mathematischen Arbeiten von László Kalmár. Mat. Lapok **6** (1955), 138–150. (Hungarian. Russian and German summaries)

List of works of László Rédei, winner of the 1955 Kossuth prize. Mat. Lapok **6** (1955), 323–327. (Hungarian)

Fuchs, László. Life and works of Tibor Szele, 1918-1955. Mat. Lapok **6** (1955), 97–129. (Hungarian. Russian and English summaries)

A list of Szele's published mathematical papers is included.

Denjoy, Arnaud. Notice nécrologique sur M. Hermann Weyl. C. R. Acad. Sci. Paris **241** (1955), 1665–1667.

See also: Künzi, p. 473.

FOUNDATIONS, MATHEMATICAL LOGIC

★ **Ferrater Mora, Jose, y Leblanc, Hugues. Lógica matemática. [Mathematical logic.]** Fondo de Cultura Economica, México-Buenos Aires, 1955. 210 pp.

This is a sketchy survey of symbolic logic, covering the logic of propositions, the first-order predicate calculus, the logic of classes and relations and the principles of the theory of types. In either field many formulas are stated, but there is no axiomatic treatment and only a rudimental attempt at systematization. Many proofs, and even definitions, are omitted; for instance, the definition of a relation as a class of pairs is not mentioned; yet the intersection of two relations is used without explanation. What is said about Brouwer's point of view is erroneous. The book can be useful for those who wish to have a general idea of the problems and methods of symbolic logic, without bothering about mathematical deductions in logic. *A. Heyting* (Amsterdam).

Feys, Robert. Boolean methods of development and interpretation. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 107–112.

Brouwer, L. E. J. The effect of intuitionism on classical algebra of logic. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 113–116.

Umezawa, Toshio. Über die Zwischensysteme der Aussagenlogik. Nagoya Math. J. **9** (1955), 181–189.

A new proof of Gödel's result that there is an infinity of systems of the propositional calculus between the classical and the intuitionistic system [Anz. Akad. Wiss. Wien Math.-Nat. Kl. **69** (1932), 65–66]. The author's systems are different from Gödel's. *A. Heyting.*

Reichbach, J. Completeness of the functional calculus of first order. Studia Logica **2** (1955), 229–250. (Russian. English summary)

Let U be the first-order functional calculus augmented by a set of formulae of this calculus. '$\vdash\alpha$' will stand for 'α is provable in U', and '$\nvdash\alpha$' for 'α is not provable in U'. U is called consistent if and only if there is an α such that $\nvdash\alpha$. The Skolem-Gödel theorem asserts that if U is consistent, then every theorem of U is satisfiable in the domain of natural numbers. Here, as often, this theorem is mislabeled as the Skolem-Löwenheim theorem [see Feferman and Tarski, J. Symb. Logic, **18** (1953), 339–340]. Gödel's theorem states that every true α is provable in the first-order functional calculus. The proofs of these theorems given by the author are related to those given by Henkin [ibid. **14** (1949), 159–166; MR **11**, 487], and by Rasiowa and Sikorski [Fund. Math. **37** (1950), 193–200; **38** (1951), 230–232; MR **12**, 661; **15**, 385]. However, it differs from both of them by avoiding all "powerful" methods of proof like a denumerable sequence of inductions in Henkin's proof. The notion of satisfaction is defined so as to establish a one-to-one correspondence between the individual free variables and a sequence of

objects, and between functional variables and a double sequence of predicates. A set I of formulae of U is called a proper prime ideal when (1) if $\alpha_1, \cdots, \alpha_m \in I$, then $\nvdash(\alpha_1 \vee \cdots \vee \alpha_m)$; (2) if $\beta \notin I$, then there are $\alpha_1, \cdots, \alpha_m \in I$ such that $\vdash(\alpha_1 \vee \cdots \vee \alpha_m \vee \beta)$; (3) if a general formula belongs to I, then a particularisation belongs to I. Two lemmas are crucial: if $\nvdash\alpha$, then there is an I such that $\alpha \in I$; For every I there is a double sequence of predicates of natural numbers such that for every α the sequence of natural numbers and the double sequence do not satisfy α if and only if $\alpha \in I$. The Skolem-Gödel theorem and Gödel completeness theorem follow easily. *H. Hiż.*

Surányi, János. On the reduction theory of the decision problem of symbolic logic. Mat. Lapok **6** (1955), 180–197. (Hungarian. Russian and English summaries)

This paper acquaints the reader with the problem in the title. After sketching simple proofs of former results, the following result is announced: Any given first-order formula is equivalent (as to satisfiability) to one of the form

$$\forall x_1 x_2 \exists x_3 \mathfrak{M}_1 \vee \forall y_1 y_2 y_3 \mathfrak{M}_2$$

where the formulae M_1 and M_2 do not contain any more quantifiers, and only a single binary predicate occurs in them besides mononary ones. *Author's summary.*

Valpola, Veli. Eine Eigenschaft gewöhnlicher negationsloser Kalküle der Propositionen- und Prädikatenlogik. Math. Scand. **3** (1955), 107–114.

Consider a formal system built up out of propositional variables, propositional constants, and connectives f_{mn} with m argument places ($m, n = 1, 2, \cdots$), and in which the following rules are valid: (i) modus ponens for f_{21}; (ii) at least one formula is derivable; (iii) every propositional constant, if any, is a derivable formula; (iv) if $s_1, s_2, \cdots, s_m$ are derivable formulas, then $f_{mn}s_1s_2\cdots s_m$ is a derivable formula. Then the following Theorem 3 holds. If s_1 and s_2 are formulas such that $f_{21}s_1s_2$ is derivable, then either some propositional variable occurs both in s_1 and s_2, or s_2 is derivable. Most of the positive propositional calculi which have been considered satisfy (i)-(iv), but Church's weak positive implicational calculus [Kontrolliertes Denken, Alber, München, 1951, pp. 22–37] does not satisfy (iv). Yet Theorem 3, and even a somewhat stronger theorem, holds for this calculus. *A. Heyting.*

Robinson, Raphael M. Primitive recursive functions. II. Proc. Amer. Math. Soc. **6** (1955), 663–666.

The Cantor pairing function

$$x = J(u, v) = \tfrac{1}{2}(u+v)\,(u+v+1)+u$$

establishes a one-to-one correspondence between the pairs of natural numbers and all natural numbers; the inverse functions are $Kx = u$, $Lx = v$. Continuing an earlier paper [Bull. Amer. Math. Soc. **53** (1947), 925–942; MR **9**, 221], the author proves that all primitive recursive functions of one variable can be obtained by starting with the two functions S (successor function) and K (or S and L), and repeatedly defining a new function Fx from known function Ax and Bx by either of the sets of formulas (i), (ii), (iii) or (i), (ii), (iv). (i) $Fx = BAx$, (ii) $Fx = B^n0$, (iii) $Fx = Ax + Bx$, (iv) $Fx = J(Ax, Bx)$. *A. Heyting* (Amsterdam).

Robinson, Julia. A note on primitive recursive functions. Proc. Amer. Math. Soc. **6** (1955), 667–670.

Using results from the paper reviewed above and from her paper on general recursive functions [same Proc. **1** (1950), 703–718; MR **12**, 469], the author proves that all primitive recursive functions of one variable can be obtained by starting with two (complicated) primitive recursive functions and repeatedly using (i) and (ii) (see the preceding review). She also shows that no single function is sufficient, even if (ii) is replaced by $Fx = B^a a$, where various values of a may be used. *A. Heyting.*

de la Cinta Badillo, M^a. Foundations in relation to polyvalent symbolic logic. Gac. Mat., Madrid (1) **7** (1955), 7–13. (Spanish)

The author gives a brief exposition of many-valued logics and suggests some interpretations. *A. Rose.*

Wang, Hao. Negative types. Quart. Rev. Psychol. Philos. (N.S.) **61** (1952), 366–368.

Untersuchung der Frage, ob in einer Erweiterung T' der (einstelligen) Typentheorie T durch eine unbegrenzte Folge von negativen Typen ein „Unendlichkeitsaxiom" (UA) beweisbar ist. Inhaltliche Argumente, deren Formalisierung die Existenz jeder endlichen Anzahl von Elementen in jeder Stufe liefert, legen das nahe; da aber (1) T' mindestens so stark ist wie T ohne UA, (2) der Widerspruchsfreiheitsbeweis durch ein endliches Modell auf T' übertragbar ist (jede T'-Herleitung enthält nur endlich viele Typen), (3) der Beweis (2) in T mit UA formalisierbar ist, kann nach Gödels „Unableitbarkeitstheorem" kein UA in T' beweisbar sein. — Vorschlag des Verf.'s, durch Kombination von T' mit seiner Syntax zu einem System zu gelangen, das ebenso oder fast ebenso stark wie T mit UA ist. *G. Hasenjaeger* (Zbl **49**, 148).

Kreisel, G., and Wang, Hao. Some applications of formalized consistency proofs. Fund. Math. **42** (1955), 101–110.

Führt man für die formalisierte Arithmetik Z_μ die Formalismen $Z^{(n)}$ dadurch ein, dass ein Beweis von $Z_\mu^{(n)}$ ein Beweis von Z_μ sein soll, der höchstens n verschiedene kritische ε-Formeln enthält, so liefert der Ackermannsche Wf. Beweis von Z_μ, dass Con $Z_\mu^{(n)}$ im Z_μ beweisbar ist. Entsprechend ist auf Grund des Wf. Beweises mit Hilfe einer Wahrheitsdefinition in Z auch Con $Z^{(n)}$ beweisbar, wenn $Z^{(n)}$ die Beweise von Z enthält, in denen höchstens n gebundene Variable vorkommen. Von diesen Sätzen werden mehrere interessante Anwendungen gegeben, u.a. die Umkehrung des Satzes von Skolem-Gödel-Bernays: Wenn ein endliches Axiomensystem F ein Modell in Z hat, dann ist Con F in Z beweisbar. *P. Lorenzen* (Bonn).

Bernays, Paul. Betrachtungen über das Vollständigkeitsaxiom und verwandte Axiome. Math. Z. **63** (1955), 219–229.

The author formulates a number of alternatives to Hilbert's completeness axiom in the theory of real numbers. Thus, it is shown that the system of real numbers can be characterised as an ordered field K such that if a is an element of any extension of K, then the set $K \cup \{a\}$ is either not densely ordered or it possesses a first or a last element. Again, the system of real numbers can be characterised also as an ordered field K which satisfies Cantor's axiom and such that any initial segment of K without a last element is co-final with a countable sequence of elements of K. Furthermore, the author shows

that, starting with the additive set of positive real numbers, one can regain the field of real numbers by means of a suitable definition of the product, which is based on the theory of proportions. The positive real numbers can be characterised accordingly as a particular kind of semi-group which is Archimedean and maximal (complete) and in which a specified element is distinguished as unity. Examples are constructed to show that the fact that a field is Archimedean cannot be expressed by means of the order type of the field alone, and that a field may satisfy Cantor's axiom with being Archimedean. *A. Robinson* (Toronto, Ont.).

Devidé, Vladimir. Ein Axiomensystem für die natürlichen Zahlen. Arch. Math. **6** (1955), 408–412.

Slight variant of Peano's axioms for the natural numbers. *G. Kreisel* (Princeton, N.J.).

Medvedev, Yu. T. Degrees of difficulty of the mass problem. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 501–504. (Russian)

Any mathematical task whose achievement can be resolved into an infinite sequence of elementary acts whose results can each be characterized by a natural number is a mass problem. Arithmetically speaking, a mass problem is the task of "constructing certain functions (with natural numbers as values and arguments) having a given property." A given mass problem A completely determines and is completely determined by its set of solving functions $\{f\}$ and we write $A=\{f\}$. The null class of functions is associated with an improper or absolutely unsolvable problem; a problem is called algorithmically solvable if at least one of its solving functions is general recursive. A problem $B=\{g\}$ reduces to a problem $A=\{f\}$ if there is a partial recursive operation $R(\phi)$ carrying each function f in A into some function g in B. Two problems are called equivalent if each reduces to the other. The collection of problems equivalent to a given problem A is called the degree of difficulty $|A|$ of A. Write $a\geqq b$ if and only if any problem with degree of difficulty b is reduced to a problem with degree of difficulty a. The collection Ω of degrees of difficulty is a complete distributive lattice for meet, $a\wedge b$, and join $a\vee b$, suitably defined. The operation $a\supset b$ is defined as the least c such that $a\wedge c\geqq b$. For a fixed segment $0\leqq x\leqq e$ of Ω we define $\bar{x}=x\supset e$. We interpret $\vdash X$ if X is in Ω to mean that the problem having degree of difficulty X is solvable. Then all axioms and rules of the intuitionistic propositional logic are satisfied by the elements of an arbitrary segment $0\leqq x\leqq e$ of Ω. This is related to a result of A. N. Kolmogorov [Math. Z. **35** (1932), 58–65]. There follows a list of results concerning problems of solvability and enumerability related to S. C. Kleene and E. L. Post, Ann. of Math., (2) **59** (1954), 379–407 [MR **15**, 772].

Errata: Page 502, line –11 should read $\vdash[(a\supset b)\wedge(b\supset c)]\supset(a\supset c)$; page 502, line –4, first formula should read $\vdash[(a\supset b)\wedge(b\supset c)]\supset(a\supset c)$; page 502, last line should read $\vdash[(a\supset b)\wedge(a\supset\bar{b})]\supset\bar{a}$. *E. J. Cogan* (Hanover, N.H.).

★ **Gonseth, Ferdinand. La géométrie et le problème de l'espace. VI. Le problème de l'espace.** Editions du Griffon, Neuchatel, 1955. 173 pp. 1170 francs.

[For parts I–V, see MR **15**, 594.] After giving a sketch of plane elliptic geometry, first as geometry in the sheaf, then treated by means of the Cayley-Klein model, the author draws his conclusions on the notion of space and, more generally, on the theory of knowledge. According to him, geometry is an open science in this sense, that the intuitions and concepts underlying it, and the methods used in it, are never definitively fixed, but that they gradually define themselves and constantly refine themselves in the progress of science. He then argues that scientific knowledge in general must be of the same kind. *A. Heyting* (Amsterdam).

★ **Barone, Francesco. Un'apertura filosofica della logica simbolica peaniana.** In memoria di Giuseppe Peano, pp. 41–50. Liceo Scientifico Statale, Cuneo, 1955.

★ **Geymonat, Ludovico. I fondamenti dell'aritmetica secondo Peano e le obiezioni "filosofiche" di B. Russell.** In memoria di Giuseppe Peano, pp. 51–63. Liceo Scientifico Statale, Cuneo, 1955.

★ **Rényi, Alfréd. Die prinzipiellen Fragen der Wahrscheinlichkeitsrechnung im Lichte des dialektischen Materialismus.** Philosophisches Jahrbuch, 1952, Zusammenfassung, pp. 7–8. Akadémiai Kiadó, Budapest, 1954.

Abstract of a paper in Mat. Lapok **1** (1949), 27–64; MR **11**, 374.

★ **Aliotta, Antonio. Valore filosofico della teoria di Albert Einstein.** Cinquant'anni di Relatività, 1905–1955, pp. 457–476. Editrice Universitaria, Firenze, 1955.

See also: Popper, p. 378; Neumann, p. 448; Robinson, p. 449; Hintikka, p. 449; Büchi and Wright, p. 455.

ALGEBRA

Neumann, B. H. An embedding theorem for algebraic systems. Proc. London Math. Soc. (3) **4** (1954), 138–153.

Let C be a class of algebraic systems (in the most general sense) and P a property that is meaningful for the systems of C. Then a system A of C is said to have the property P "locally" if every finitely generated subsystem of A has the property P; and "the local theorem" is said to hold in C for the property P (or P is a property "of local character" for C) if a system A of C has the property P if and only if it has the property P locally. A number of such local theorems are known, especially in the theory of groups. The author proves a very general local theorem about the embeddability of a partial algebraic system of a class in a full system (that is, one in which the operations can be carried out without restriction). To lend precision to this somewhat vague statement, an apparatus of mixed logical and algebraic nature to deal with embedding questions in the most general algebraic systems is very lucidly and carefully developed. The theorem is then proved and applied to yield among its many corollaries the following cases: A group G can be fully ordered if (and only if) every finitely generated subgroup of G can be fully ordered. A ring R can be embedded in a division ring if every finitely generated subring of R can be so embedded. The edges of a graph can be coloured with k colours so that no concurrent edges are equally coloured if the edges of every finite subgraph can be so coloured.

[For this last result see also de Bruijn and Erdös, Indag. Math. 13 (1951), 369–373; MR 13, 763.]

K. A. Hirsch (Zbl 56, 29).

Robinson, A. Note on an embedding theorem for algebraic systems. J. London Math. Soc. 30 (1955), 249–252.

Eine Gruppe lässt sich bekanntlich genau dann total ordnen, wenn für je endlich viele Elemente $a_1, \cdots, a_n$ die 2^n aus $a_1^{\varepsilon_1}, \cdots, a_n^{\varepsilon_n}$ $(\varepsilon_\nu = \pm 1)$ erzeugten invarianten Unterhalbgruppen nur das Einselement gemeinsam haben [Math. Z. 52, 483–526 (1949); MR 11, 497]. Als Folgerung ergibt sich, dass sich eine Gruppe total ordnen lässt, wenn jede endlich erzeugbare Untergruppe total ordenbar ist. Diese Folgerung ist von B. H. Neumann in der oben referierten Arbeit aus einem allgemeinen Einbettungssatz abgeleitet worden. Verf. zeigt nun, wie sich der Neumannsche Einbettungssatz und diese Folgerung unmittelbar aus dem metamathematischen Vollständigkeitsatz der elementaren Prädikatenlogik gewinnen lassen.

P. Lorenzen (Bonn).

Hintikka, K. Jaakko. An application of logic to algebra. Math. Scand. 2 (1954), 243–246.

Relying on the theorem, stated by R. P. Dilworth [Bull. Amer. Math. Soc. 53 (1947), 906], that every finite lattice is isomorphic to a sublattice of a semi-modular lattice, the author shows that every lattice is isomorphic to a sublattice of a semi-modular lattice. For the proof, use is made of the extended completeness theorem of the lower predicate calculus and of the technique of "complete diagrams". The metamathematical argument is summed up in the following more general principle. Let Φ and Ω be two sets of axioms in the lower predicate calculus such that (i) every finite substructure of a model of Φ can be embedded in a finite model of Φ and (ii) every finite model of Φ can be embedded in a model of Φ which is also a model of Ω. Then every model of Φ can be embedded in a model of Φ which is at the same time a model of Ω. [For related principles, cf. the two papers reviewed above.]

A. Robinson (Toronto, Ont.).

Combinatorial Analysis

Sade, A. Sur le problème des transvasements. J. Reine Angew. Math. 195 (1956), 1–2 (1955).

Three vases of contents x, p, q, respectively, where $x > p > q$ and $(p, q) = 1$ contain between themselves s units of liquid $x > s > p$. The liquid may be transferred from one vase to another without any measuring device in such a way that the contents of the vases are always always known. The author finds necessary and sufficient conditions that one may pass from an initial state in which the second vase contains i units of liquid to a state in which it contains j units. The necessary number of steps can also be computed. An example where p, q, i and j are all larger than 10^5 is calculated in the paper with great ease. This shows that the author's criterion is quite practicable and does not lead to excessive calculations.

H. B. Mann (Columbus, Ohio).

Parker, E. T., and Mood, A. M. Some balanced Howell rotations for duplicate bridge sessions. Amer. Math. Monthly 62 (1955), 714–716.

Howell rotations have the following properties: (a) Every partnership plays once at a table with every other partnership. (More than one board may be played at the table.) (b) Every partnership plays every board once. A balanced rotation has the additional property: (c) Every partnership competes (in the sense of playing the same hands) equally often with every other partnership. Because every partnership plays once at a table with every other partnership, the minimum number of rounds for T tables is $2T-1$.

The authors show that a balanced rotation is impossible in $2T-1$ rounds when T is odd, and exhibit for the first time balanced rotations for 5, 6, 7 and 8 tables. The rotations for 6 and 8 tables use only the minimum number of rounds, but those for 5 and 7 tables use twice the minimum number. No general method of construction is known to the authors. For completeness, rotations are presented for 2, 3 and 4 tables.

W. S. Connor.

Good, I. J. On the marking of chess-players. Math. Gaz. 39 (1955), 292–296.

Binet, F. E., Leslie, R. T., Weiner, S., and Anderson, R. L. Analysis of confounded factorial experiments in single replications. North Carolina Agricultural Experiment Station, Tech. Bul. No. 113 (1955), 64 pp.

The authors utilize a system of orthogonal linear forms introduced by Yates and modified by Bainbridge. This is obtained from the assumption that the means of the observations are given by polynomials in the levels of the factors. This enables one to define the degree of an interaction term. The authors' object is to give designs which confound interactions of high degree rather than of high order. The two concepts coincide of course in the case of a 2^n design. With this object in mind the authors give explicitly many plans of designs with number of factors and levels within ranges likely to be of use. Many examples of experiments arranged in these layouts are given and their analysis carried through.

H. B. Mann.

Turán, P. On a problem in the theory of determinants. Acta Math. Sinica 5 (1955), 411–423. (Chinese. English summary)

This paper is concerned with Sylvester's problem to determine all values of n for which a matrix of ± 1's exists in which any two rows are orthogonal. The known results on this problem are reviewed and it is linked up with Hadamard's problem concerning the absolute maximum M_n of determinants with real elements of absolute value not exceeding 1. The equality sign in $M_n \leqq n^{n/2}$ is only attained if any two rows are orthogonal. The constants M_n themselves can be expressed as $M_n = \lim_{k\to\infty} M_n^{(2k)}$, where $2^n M_n^{(2k)}$ is the $2k$th power mean of all determinants of order n formed with elements ± 1. G. Szekeres and the author [Math. Naturwiss. Anz. Ungar. Akad. Wiss. 56 (1937), 796–806] evaluated $M_n^{(2)}$ as $(n!)^{\frac{1}{2}}$ and $M_n^{(4)} = (n!)^{\frac{1}{2}}(\psi(n))^{\frac{1}{2}}$, where $\psi(1)=1$, $\psi(2)=2$ and $\psi(n)=\psi(n-1)+2n^{-1}\psi(n-2)$, which leads to $M_n > (n+4)^{\frac{1}{4}} n!/\sqrt{5}$. A simple method for the determination of $M_n^{(4)}$ is now obtained which the author expects to be also effective for $M_n^{(6)}$. [For related work see also P. Turán, ibid. 59 (1940), 95–105; MR 2, 341.]

O. Taussky-Todd (Washington, D.C.).

See also: Castrucci, p. 522.

Linear Algebra, Polynomials, Invariants

Gonçalves, J. Vicente. Démonstration du théorème de Hamilton-Cayley. Univ. Lisboa. Revista Fac. Ci. A. (2) 3 (1954–1955), 330.

Gonçalves, J. Vicente. Démonstration du théorème de Binet-Cauchy. Univ. Lisboa. Revista Fac. Ci. A. (2) 3 (1954–1955), 327–329.

Wielandt, Helmut. On eigenvalues of sums of normal matrices. Pacific J. Math. 5 (1955), 633–638.

A theorem by Bendixson [Acta. Math. 25 (1902), 359–365] states that the eigenvalues of a matrix $X+iY$, where X, Y are hermitian, are contained in the rectangle formed by the points $\xi_1+i\eta_1$, $\xi_1+i\eta_n$, $\xi_n+i\eta_1$, $\xi_n+i\eta_n$, where ξ_1, ξ_n (resp. η_1, η_n) are the smallest and largest eigenvalues of X (resp. Y). For $n=2$ Parker [Amer. Math. Monthly 60 (1953), 247–250; MR 14, 836] showed that the set of eigenvalues of all X_1+iY_1, where X_1, Y_1 are hermitian and have the same eigenvalues as X, Y is the intersection of the rectangular hyperbola passing through the vertices of this rectangle and the rectangle itself. The sum of any two normal matrices with fixed eigenvalues $\{\alpha_\nu\}$ and $\{\beta_\nu\}$ is now considered and earlier results of the author on inclusions of eigenvalues of normal matrices are used [Math. Ann. 124 (1949), 234–241; MR 11, 307]. It is shown that the set Λ of all possible eigenvalues of the sum is the intersection of the sets $(\{\alpha_\nu\}+\Gamma)$ where Γ runs through all "circular" sets $a\xi+b\eta+c(\xi^2+\eta^2)+d\geqq 0$ which contain the set $\{\beta_\nu\}$. In Bendixson's case Λ can be expressed as the intersection of all "hyperbolic" sets $a\xi+b\eta+c(\xi^2-\eta^2)+d\geqq 0$ which contain $\{\alpha_\nu\}+\{\beta_\nu\}$. This contains Parker's result. It is shown that only the distinct α_ν and β_ν matter and that Λ contains the corresponding set for a subset of the α_ν and β_ν. Further, Λ is a closed bounded set and in Bendixson's case every connected component of it is simply connected.

O. Taussky-Todd (Washington, D.C.).

Parodi, Maurice. Sur un critère de stabilité de fonctionnement des machines mathématiques analogiques. C. R. Acad. Sci. Paris 241 (1955), 1104–1105.

It is shown that the eigenvalues of a $n\times n$ matrix (a_{ik}) have positive real parts if 1) all elements a_{ik} are real; 2) all diagonal elements a_{ii} are positive, 3) n positive numbers $K_1, K_2, \cdots, K_n$ can be found such that $\sum_{i=1}^{n}(1+K_i)^{-1}\leqq 1$ and $a_{ii}>K_i m_i$ for all values of i with $m_i=\max_{k\neq i}|a_{ik}|$. *H. Bückner* (Schenectady, N.Y.).

See also: Turán, p. 449; Specht, p. 466; Remez, p. 477; Hajnal, p. 501; Aparo, p. 536; Penrose, p. 536; Householder, p. 536; Gregory, p. 537.

Lattices

Szász, Gábor. Remarks on weakly complemented lattices. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 451–456. (Hungarian)

All lattices discussed here have a least element 0. A lattice is weakly complemented if to any two distinct elements u and v there corresponds an element x such that $x\cap(u\cap v)=0$ and $x\cap(u\cup v)\neq 0$. A lattice is semi-complemented if to every element u (except possibly the greatest element, in case there is one) there corresponds a non-zero element x (a semi-complement of u) such that $x\cap u=0$. Theorem: A lattice is weakly complemented if and only if to any two distinct elements u and v there corresponds an element x that is a semi-complement of u but not a semi-complement of v. A weakly complemented lattice need not be complemented, and a complemented lattice need not be weakly complemented. Theorem: A relatively complemented lattice is weakly complemented. Theorem: A weakly complemented lattice is semi-complemented. *P. R. Halmos* (Chicago, Ill.).

Dwinger, Ph. On the group of automorphisms of the lattice of closure operators of a complete lattice. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17 (1955), 507–511.

By a closure operator over a complete lattice L is meant a function φ on L to L such that $a\leqq b$ implies that $a\leqq\varphi(a)\leqq\varphi(b)=\varphi^2(b)$. The set C_L of all closure operators over L is a complete lattice under the natural inclusion relation. It is shown that L and C_L have isomorphic groups of automorphisms. *B. Jónsson.*

Fujiwara, Tsuyoshi. Remarks on the Jordan-Hölder-Schreier theorem. Proc. Japan Acad. 31, 137–140 (1955).

Generalizing slightly the Jordan-Hölder-Schreier theorem for weakly associable congruence relations over an algebra, the author uses this result to give a new proof of the Jordan-Hölder-Schreier theorem for modular lattices. *B. Jónsson* (Berkeley, Calif.).

Floyd, E. E. Boolean algebras with pathological order topologies. Pacific J. Math. 5 (1955), 687–689.

A topology on a complete lattice is said to be σ-consistent (with the partial ordering) if every monotonic sequence converges to its bound. Let L be the lattice of all regular open subsets of the unit interval. Results: I. L has no σ-consistent Hausdorff topology. II. There exists a double sequence of elements $X_{i,j}$ of L such that each row order converges to ϕ, while no sequence with terms $X_{i,j(i)}$ order converges to ϕ. III. The operation of subtraction in the vector lattice of all continuous real-valued functions on the Stone space of L is not continuous in any σ-compatible Hausdorff topology. The last result shows that Th. 4 on p. 242 of Birkhoff's "Lattice theory" [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; MR 10, 673] is false, while II solves Birkhoff's Problem 77. *B. Jónsson* (Berkeley, Calif.).

Ribeiro, Hugo. Topological groups and Boolean algebras with operators. Univ. Lisboa. Revista Fac. Ci. A. (2) 4, 195–200 (1955).

With the obvious interpretation of the symbols involved, the subsets of a topological group constitute a system $\langle A, \cup, 0, \cap, 1, \cdot, e, {}^{-1}, \sim\rangle$ such that (i) $\langle A, \cup, 0, \cap, 1, \cdot, e, {}^{-1}\rangle$ is an integral relation algebra and (ii) $\langle A, \cup, 0, \cap, 1, \sim\rangle$ is a closure algebra. It is noted that the continuity of the group operations implies that (iii) $(\bar{x})^{-1}=(\widetilde{x^{-1}})$ and that (iv) if $(x\cdot y)\cap\bar{z}\neq 0$, then there exist u, v such that $x\cap\bar{u}\neq 0\neq y\cap\bar{v}$ and $\bar{u}\cdot\bar{v}\subseteq\bar{z}$. Certain elementary results about systems satisfying (i)–(iv) are derived, corresponding to well-known properties of topological groups, such as the fact that the closure of a subgroup is a subgroup. *B. Jónsson* (Berkeley, Calif.).

See also: Hintákka, p. 449; Šik, p. 456; Nikodým, p. 468; Haupt and Pauc, p. 469; Cristescu, p. 510.

Rings, Fields, Algebras

★ **Séminaire P. Dubreil de la Faculté des Sciences de Paris, 1954/1955. Algèbre et théorie des nombres.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955. iii+166 pp. (polycopiées).

These are paper-bound copies of articles presented at P. Dubreil's seminar on Algebra and the Theory of Numbers at the Faculté des Sciences de Paris during the academic year 1954–55. Most of these have appeared or will appear separately. Of the twenty-seven sessions of the seminar, this book contains the papers given at thirteen of them. Thirteen other sessions were devoted to a study of the theory of ideals and applications to algebraic geometry. The text for the ninth session, by J. Riguet on "Méthode de denombrement de Pólya et théorie des caractères" is missing.

The material in this book consists of the following. 1) R. Descombes on non-homogeneous Diophantine approximations [Bull. Soc. Math. France **82** (1954), 197–299; MR **16**, 803]. 3) M. Krasner on non-existence of certain field extensions [C.R. Acad. Sci. Paris **237** (1953), 370–372, 685–687; MR **15**, 96, 97]. 5 and 7) M. Lazard on formal Lie groups with one parameter [ibid. **239** (1954), 942–945; MR **16**, 219]. 11) L. Lesieur on irreducible ideals in a ring or commutative semi-group, studied in terms of lattice theory. 13) P. Samuel on plane curves over fields of characteristic 2. 14) J. Guérindon on a generalization of ideal theory, extending the Noether theory to groups with operators. 15) C. Pisot on algebraic integers with a single conjugate greater than unity in absolute value. [with J. Dufresnoy, Ann. Sci. Ecole Norm. Sup. (3) **70** (1953), 105–133; Bull. Sci. Math. (2) **77** (1953), 129–136; C.R. Acad. Sci. Paris **238** (1954), 1551–1553; MR **15**, 605, 856]. 17) F. Chatelet on splitting fields of central simple algebras, treated as a diophantine problem on algebraic curves. 19) I. Fleischer on homomorphisms in universal algebras. 21–23) J. Petresco on free groups and various ways of choosing free generators of subgroups. 25) R. Croisot on lattice properties of ideals in general rings and semi-groups. 25) P. Jaffard on partially ordered abelian groups and their embedding in direct products of simply ordered groups [cf. J. Math. Pures Appl. (9) **32** (1953), 203–280; MR **15**, 284]. *Marshall Hall, Jr.*

Hughes, D. R. Planar division neo-rings. Trans. Amer. Math. Soc. **80** (1955), 502–527.

A division neo-ring (DNR) is the non-associative analogue of a division ring in the sense that the additive (+) and multiplicative (·) systems are loops rather than groups. A planar division neo-ring (PDNR) is a DNR that may be used to coordinatize an affine plane by identifying planar ternary rings $F(a, b, c)$ with the DNR operation $a \cdot b + c$. [See also M. Hall, same Trans. **54** (1943), 229–277; MR **5**, 72.]

Algebraic properties of finite planar ternary rings are developed and the relationships between the affine planes, the planar ternary rings and the complete sets of orthogonal latin squares usually associated with the affine planes are determined.

The author considers the structure of DNRs and PDNRs and obtains general structure theorems of which the following is typical: "The center of an arbitrary PDNR is a PDNR." With the restriction that the PDNR be finite and have associative multiplication, the concept of difference sets [see also Bruck, ibid. **78** (1955), 464–481; MR **16**, 1081] is introduced and the following principal result is obtained: "If $(R, +, \cdot)$ is a finite PDNR of order n with commutative and associative multiplication, and if p is a prime dividing n, then the mapping $x \to x^p$ is an automorphism of $(R, +, \cdot)$. Principally by the use of this result it is shown that all PDNRs with associative and commutative multiplication of order ≤ 250 are shown to have prime power order. Although the author was able to find no general proof he points out that all PDNRs with associative (even power-associative) multiplication of order ≤ 250 are also commutative and hence of prime power order. The paper concludes with examples of infinite PDNRs. *L. J. Paige* (Los Angeles, Calif.).

San Soucie, R. L. Right alternative rings of characteristic two. Proc. Amer. Math. Soc. **6** (1955), 716–719.

A right alternative division ring will be alternative if its characteristic is not two, but for characteristic two there exist right alternative division rings which are not alternative. The author has shown [same Proc. **6** (1955), 291–296; MR **16**, 896] that a right alternative ring of characteristic two is alternative if (and only if) it satisfies the Moufang identity $w(xy \cdot x) = (wx \cdot y)x$. For a general right alternative ring R of characteristic not two, Kleinfeld [ibid. **4** (1953), 939–944; MR **15**, 595] has shown that R will be alternative if and only if it has property (P) which asserts that $(t, u, v)^2 = 0$ implies $(t, u, v) = 0$ whenever the elements t, u, v in this associator are an "alternative triple" as defined by Kleinfeld. This paper combines these two results to show that if R is of characteristic two, it will be alternative if (and only if) property (P) and the Moufang identity hold.

Marshall Hall, Jr. (Columbus, Ohio).

Hughes, D. R. Additive and multiplicative loops of planar ternary rings. Proc. Amer. Math. Soc. **6** (1955), 973–980.

Sei R eine nichtleere Menge von Elementen, unter denen 0 und 1 vorkommt. Eine für alle Elemente aus R erklärte Operation $F(a, b, c) \in R$ heisst planarer ternärer Ring über R, wenn gilt: $F(a, 0, c) = F(0, b, c) = c$ für alle $a, b, c \in R$; $F(a, 1, 0) = F(1, a, 0) = a$ für alle $a \in R$; $F(x, a, b) = F(x, c, d)$ ist eindeutig lösbar in R für alle $a, b, c, d \in R$ mit $a \neq c$; $F(a, b, x) = c$ ist in R eindeutig lösbar für alle $a, b, c \in R$; $F(a, x, y) = b$, $F(c, x, y) = d$ ist eindeutig lösbar in R durch ein geordnetes Paar x, y für beliebige $a, b, c, d \in R$ mit $a \neq c$. Durch $F(1, a, b) = a + b$ und $F(a, b, 0) = ab$ sind zwei als Addition und Multiplikation definierte Operationen in R erklärt. Mit diesen binären Operationen als Verknüpfung bilden die Elemente von R und die von 0 verschiedenen Elemente von R je eine loop mit dem neutralen Element 0 resp. 1. Diese loops heissen additive und commutative loops des planaren ternären Rings. Die Frage, ob diese loops ausgezeichnet sind, verneint Verf. durch den Beweis des Satzes: Ist R die Menge der nichtnegativen ganzen Zahlen, R^* die Menge der positiven ganzen Zahlen, und ist in R eine additive loop, in R^* eine multiplicative loop gegeben, so existiert ein planarer ternärer Ring $F(a, b, c)$, dessen additive und multiplicative loop mit den gegebenen loops übereinstimmen. Ferner gilt: Zu einer gegebenen abzählbar unendlichen additiven Gruppe G existiert ein planarer ternärer Ring, dessen additive loop zu G isomorph ist und der zusätzlich linear ist, d.h. es gilt $F(a, b, c) = ab + c$ für alle $a, b, c \in G$; die Frage bleibt offen, ob die letzte Aussage auch noch gilt, wenn G nur abzählbar unendliche loop ist.

R. Moufang (Frankfurt a.M.).

Foster, Alfred L. The identities of — and unique subdirect factorization within — classes of universal algebras. Math. Z. **62** (1955), 171–188.

An algebra $\mathfrak{P}=(P, \sigma, \cdots)$ with at least two elements is called primal if every function on P^n to P (n finite) can be obtained by superposition from the primitive operations and the identity functions. A set $\{\mathfrak{A}_1, \cdots, \mathfrak{A}_k\}$ of algebras (of a common species) is said to be independent if, given any expressions (algebraic functions) $\Phi_1, \cdots, \Phi_k$, there exists an expression Φ which agrees with Φ_i on $\mathfrak{A}_i$. A set of primal algebras is called a primal cluster if every finite subset is independent. The principal result states that, given any finite subset $\tilde{\mathfrak{Q}}$ of a primal cluster $\tilde{\mathfrak{P}}$, every algebra $\mathfrak{P}$ which satisfies all the identities of $\tilde{\mathfrak{Q}}$ is isomorphic to a direct product of subdirect powers of algebras in $\tilde{\mathfrak{Q}}$. If $\mathfrak{P}$ is finite, "subdirect powers" can be replaced by "direct powers". It is also shown that if two finite subsets of a primal cluster have the same identities, then the two sets are equal. *B. Jónsson.*

Recillas, J., F., and Lluis R., E. Hilbert's function in semilocal rings. Bol. Soc. Mat. Mexicana **11** (1954), 1–8. (Spanish)

Soient $\mathfrak{o}$ un anneau semi local, $\mathfrak{m}_1, \cdots, \mathfrak{m}_h$ ses idéaux maximaux, $\mathfrak{a}$ un idéal ouvert de $\mathfrak{o}$; on a $\mathfrak{a}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_h$ où $\mathfrak{q}_i$ est un idéal primaire pour $\mathfrak{m}_i$ (ou est égal à $\mathfrak{o}$). La fonction caractéristique de $\mathfrak{a}$ est la somme des fonctions caractéristiques des idéaux $\mathfrak{q}_i$, et aussi de celles des idéaux $\mathfrak{a}\cdot\mathfrak{o}_{\mathfrak{m}_i}=\mathfrak{q}_i\cdot\mathfrak{o}_{\mathfrak{m}_i}$. Etude du passage au complété de $\mathfrak{o}$.
P. Samuel (Clermont-Ferrand).

Lluis Riera, Emilio. On the open ideals in Zariski rings. Bol. Soc. Math. Mexicana **11** (1954), 33–34. (Spanish)

Soit $\mathfrak{o}$ un anneau de Zariski, c'est à dire un anneau noethérien muni de la structure définie par un idéal $\mathfrak{m}$ de $\mathfrak{o}$ tel que tout élément de $1+\mathfrak{m}$ est inversible; soit $\hat{\mathfrak{o}}$ le complété de $\mathfrak{o}$. L'application $\mathfrak{a}\to\hat{\mathfrak{o}}\cdot\mathfrak{a}$ induit une correspondance biunivoque entre l'ensemble des idéaux ouverts de $\mathfrak{o}$ et l'ensemble des idéaux ouverts de $\hat{\mathfrak{o}}$. *P. Samuel.*

Nagata, Masayoshi. An example of normal local ring which is analytically ramified. Nagoya Math. J. **9** (1955), 111–113.

Soient k un corps parfait de caractéristique 2, (u_i) et (v_i) deux suites infinies d'éléments algébriquement indépendants sur k, K le corps $k(u_1, v_1, \cdots, u_n, v_n, \cdots)$, $\mathfrak{r}$ l'anneau de séries formelles $K[[x, y]]$, $\mathfrak{o}$ l'anneau local $K^2[[x, y]][K]$ [qui est un anneau local régulier dont le complété est $\mathfrak{r}$; cf. Nagata, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28** (1954), 121–124; MR **16**, 107]. Soient c la série formelle $c=\sum_{n=1}^{\infty}(u_n x^n+v_n y^n)$ et $\mathfrak{s}$ l'anneau $\mathfrak{o}[c]$. On montre que $\mathfrak{s}$ est un anneau local noethérien et normal (c.à.d. intégralement clos), isomorphe à $\mathfrak{o}[X]/(X^2-c^2)$, dont le complété $\mathfrak{r}[X]/(X^2-c^2)$ a des éléments nilpotents non nuls (par exemple la classe de $X-c$). C'est là une circonstance qui ne peut pas se produire dans le cas d'un anneau local géométrique normal R, puisque le complété de R est un anneau d'intégrité normal [cf. Zariski, Ann. Inst. Fourier, Grenoble **2** (1950), 161–164; MR **13**, 579].
P. Samuel (Clermont-Ferrand).

Divinsky, Nathan. On commuting automorphisms of rings. Trans. Roy. Soc. Canada. Sect. III. (3) **49** (1955), 19–22.

Let T be an automorphism of an associative ring A such that $x\cdot x^T=x^T\cdot x$ for every x in A. If A is semi-simple and satisfies the descending chain condition on right ideals, then T agrees with the identity map in the simple ideals of A which are not commutative. *M. F. Smiley.*

★ **Almeida Costa, A. Anéis associativos não comutativos. [Associative noncommutative rings.]** Memórias e Estudos do Centro de Matemáticas Aplicadas ao Estudo de Energia Nuclear no. 3. Faculdade de Ciências de Lisboa, Lisboa, 1955. iv+313 pp.

This book is a continuation of a previous work published in 1948 [MR **10**, 349] and is a collection of various chapters on the theory of associative rings, most of which seem to have been published previously under other titles in Portuguese journals. The titles of the chapters are as follows. XIII: Nil subrings; XIV: The Jacobson radical; XV: On transporter and annihilator ideals in the general theory of modules; XVI: The Brown-McCoy radical, the anti-radical and the maximal regular ideal in a ring; XVII: Primitive rings; XVIII: Subdirect sums of rings, semi-simple rings; XIX: Subdirect sums of modules, semi-simple modules; XX: Modules and rings with operators, simple rings and simple algebras. The book suffers from an almost complete lack of organization; the chapters seem to follow each other at random, and even within each chapter, the connection between the various paragraphs may be very tenuous; the same topics are sometimes taken up in three or four different contexts, and it would be very hard for somebody who has no previous knowledge of the theory to get from this book a precise idea of its present status. In short, this can hardly be called a didactic work, but would be more accurately described as a collection of disconnected paraphrases on the various papers which have been published on the theory of rings in the last 20 years. *J. Dieudonné.*

Tominaga, Hisao. Supplement to my previous paper, "On primary ideal decompositions in non-commutative rings". Math. J. Okayama Univ. **3** (1954), 135–138.

In der Arbeit [dasselbe J. **3**, (1953) 39–46; MR **15**, 676] die hier ergänzt wird, hatte Verf. notwendige und hinreichende Bedingungen dafür angegeben, daß in einem beliebigen nichtkommutativen Ring R jedes Ideal als Durchschnitt endlich vieler s-Primärideale darstellbar ist. Im Supplement wird gezeigt, daß diese Bedingungen auf die Gestalt gebracht werden können, die im Referat der ursprünglichen Arbeit bereits besprochen wurde. Auch wird bewiesen, daß sich die im Referat mit (2) bezeichneten Bedingungen durch einen „Kettensatz für Rechtsquotienten" ersetzen lassen. Schließlich wird auf die Möglichkeit hingewiesen, unter Benützung gewisser Überlegungen von M. Rabin [C.R. Acad. Sci. Paris **237** (1953), 544–545; MR **15**, 96] die Bedingung (4) des Referats durch eine formell einfachere Bedingung zu ersetzen. *W. Krull* (Zbl **55**, 261).

Tominaga, Hisao. Some remarks on radical ideals. Math. J. Okayama Univ. **3** (1954), 139–142.

Unter dem Radikal $\mathfrak{r}$ des Ideals $\mathfrak{a}$ in einem i.a. nichtkommutativen Ringe R versteht Verf. den Durchschnitt aller minimalen Primoberideale von $\mathfrak{a}$. Er bespricht zunächst einige äquivalente Radikaldefinitionen und beweist dann die folgenden Sätze. 1. Die Menge aller Radikale (d.h. aller mit ihrem Radikal zusammenfallenden Ideale) bildet einen distributiven Verband, wenn man den Durchschnitt zweier Radikale $\mathfrak{r}_1$, $\mathfrak{r}_2$ wie üblich definiert und als Vereinigung von $\mathfrak{r}_1$ und $\mathfrak{r}_2$ das Radikal der Idealsumme $\mathfrak{r}_1+\mathfrak{r}_2$ einführt. 2. In R gilt die Maximalbedingung

für Radikale dann und nur dann, wenn gleichzeitig die folgenden Bedingungen erfüllt sind: a) Es gilt in R die Maximalbedingung für Primideale. b) Jedes Radikal ist in R der Durchschnitt von endlich vielen Primidealen.

W. Krull (Zbl **55**, 261).

Eilenberg, Samuel, and Nakayama, Tadasi. On the dimension of modules and algebras. II. Frobenius algebras and quasi-Frobenius rings. Nagoya Math. J. **9** (1955), 1–16.

[For part I see Nakayama, same J. **3** (1951), 139–140; MR **13**, 313.] Generalizing the usual notion, the authors define a Frobenius algebra L over the commutative ring K (with 1) as a K-projective finitely K-generated algebra with 1 for which there exists a left L-module isomorphism of L onto $\mathrm{Hom}_K(L, K)$. If there exists a two sided L-module isomorphism of L onto $\mathrm{Hom}_K(L, K)$, L is called a symmetric K-algebra. A finite-dimensional semisimple algebra over a field K, a full matrix algebra over a commutative ring K, and the group algebra of a finite group over a commutative ring K are symmetric K-algebras.

If L is a Frobenius algebra over K, and A is an L-module of finite projective dimension, then this projective dimension of A as an L-module coincides with its projective dimension as a K-module, and the global dimension of L is either 0 or ∞.

If L is the group algebra of a finite group of order r, then the global dimension of L coincides with the projective dimension of K as an L-module (the group operating trivially), and is 0 or ∞ according to whether $rK=K$ or $rK\neq K$.

If a ring L satisfies the maximal condition for left ideals and is self-injective (i.e., injective as a left L-module), then every L-module of finite projective dimension is both projective and injective. In particular, it follows again that the global dimension of L is either 0 or ∞. If K is self-injective then every Frobenius algebra over K is self-injective, and a symmetric K-algebra is self-injective if and only if K is self-injective.

A ring L is called a quasi-Frobenius ring if it satisfies the minimal conditions for left and for right ideals, if the left annihilator of the right annihilator of every left ideal coincides with that ideal, and if the same holds with the words "left" and "right" interchanged. It is shown that such a ring is self-injective, and conversely, that a self-injective ring satisfying either both maximal conditions or one minimal condition is a quasi-Frobenius ring.

G. P. Hochschild (Berkeley, Calif.).

Jennings, S. A. Radical rings with nilpotent associated groups. Trans. Roy. Soc. Canada. Sect. III. (3) **49** (1955), 31–38.

An associative ring R is called a radical ring (in the sense of Jacobson) if for every element $x \in R$ there exists an element $x' \in R$ such that $x+x'+xx'=0$. The elements of a radical ring R forms a group $\mathfrak{G}$ under the operation $x*y=x+y+xy$. The elements of R form also a Lie ring Λ, in which the commutator $x\circ y$ of x and y is defined by $x\circ y=xy-yx$. The main result of the paper is the following theorem. If the group $\mathfrak{G}$ associated with a radical ring R is nilpotent (of finite class), then the Lie ring Λ associated with R is also nilpotent and conversely, if Λ is a nilpotent Lie ring, then $\mathfrak{G}$ is a nilpotent group. The proof is selfcontained, and consists of a detailed analysis of the identities which relate group-theoretic commutators in $\mathfrak{G}$ with the Lie commutators in Λ. The author suggests that the class of the group associated with a radical ring is always equal to the class of the associated Lie ring, provided that both are finite, and proves this assertion in case R is a nil algebra over a field of characteristic zero, basing his argument on the Hausdorff-Campbell formula [Hausdorff, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. **58** (1906), 19–48].

C. W. Curtis.

Rédei, Ladislaus. Über die Ringe mit gegebenem Modul. Acta Sci. Math. Szeged **15** (1954), 251–254.

This paper contains a generalization of a theorem of R. A. Beaumont [Duke Math. J. **15** (1948), 367–369; MR **10**, 10] and of a well-known theorem about the multiplication constants of a linear associative algebra. If M is a left R-module over a ring R with unit element, we take a set of linear generators ω_i of M, where i runs through an index set I. This means that every element of M may be written in the form $\sum_i a_i\omega_i$ (all summations are extended over I) with $a_i \in R$ and $a_i=0$ for all $i \in I$ with a finite number of exceptions. The following theorem is proved. If we set $\omega_i\omega_j=\sum_k c_{ijk}\omega_k$ ($c_{ijk} \in R$; for all $i, j \in I$ we have $c_{ijk}=0$ for all $k \in I$ with a finite number of exceptions), a multiplication in M is defined which makes M a ring if and only if $\sum_{i,j}(c_{rsi}c_{ijt}-c_{sti}c_{rij})\omega_j=0$ for all $r, s, t \in I$ and $\sum_{i,j}a_ic_{ijr}\omega_j=\sum_{i,j}a_ic_{rij}\omega_j=0$ for all $r \in I$ and for all $\{a_i\}$ with $\sum_i a_i\omega_i=0$. Moreover, the ring satisfies the usual operator property $a(\alpha\beta)=(a\alpha)\beta=\alpha(a\beta)$ ($a \in R$; $\alpha, \beta \in M$), if and only if $\sum_i(ab-ba)c_{rsi}\omega_i=0$ for all $r, s \in I$ and for all $a, b \in R$. As applications of this theorem the above-mentioned property of multiplication constants and the theorem of Beaumont are given.

W. Peremans (Zbl **56**, 264).

Wolk, E. S. Free ideals in rings of functions. Proc. Amer. Math. Soc. **6** (1955), 711–715.

Let A be the ring of all functions from an infinite set X (no topology) to a ring R with unit e. An ideal I of A is free if for each x in X there exists f in I such that $f(x)=e$. Extending a theorem of Hewitt [Trans. Amer. Math. Soc. **64** (1948), 45–99, theorem 36; MR **10**, 126], the author shows that if every left ideal of R is principal, then the free left ideals of A take the form

$$\mathbf{U}_{\mathfrak{p},\alpha}\{f \in A : f(x) \in \mathfrak{p}(x) \text{ for all but a finite number of } x \in \alpha\},$$

where α runs through a dual ideal in the lattice of subsets of X, and for each x, $\mathfrak{p}(x)$ runs through an ideal in the lattice of left ideals of R.

M. Jerison (Lafayette, Ind.).

Higman, Graham. On a conjecture of Nagata. Proc. Cambridge Philos. Soc. **52** (1956), 1–4.

Nagata's theorem to the effect that any nil algebra of bounded index over a field of characteristic 0 is nilpotent is given a simple and direct proof which also guarantees the same conclusion if the characteristic exceeds the index of nilpotency. Suppose F is a free nil algebra of index n with sufficiently many generators and that $f(n)$ is the first vanishing power of F. The author has a sharper upper bound for $f(n)$ and an asymptotic lower bound.

W. G. Lister (Providence, R.I.).

Henze, Ernst. Untersuchung einer speziellen Transformation des Fernschreib-Alphabets als Transformation von Vektoren. Arch. Elek. Übertr. **9** (1955), 528–532.

The symbols of the teleprinter alphabet can be considered as vectors over the mod 2 field, GF(2). Due to the way teleprinters are made the dimension is 5, which gives 32 vectors. In this paper a special linear transfor-

mation (mod 2) of these vectors is considered in detail. This transformation is of period 8, has the unique eigenvector (1, 0, 0, 0, 0) with eigenvalue 1. Simple relay equipment to realize this transformation and its inverse are described, using Hilbert-Ackermann symbols. The author's interest in this particular matrix is merely as an example of the class of available transformations.

H. Campaigne (Washington, D.C.).

Mostowski, Andrzej. Eine Verallgemeinerung eines Satzes von M. Deuring. Acta Sci. Math. Szeged **16** (1955), 197–203.

Assume that L is a normal and algebraic extension of the (commutative) field K and that the characteristic of L is either 0 or prime to all the finite degrees $[A:B]$ for A and BCA fields between K and L. The Galois group G of L over K is a topological group which acts in a natural way on the following two K-modules: firstly L considered (not as a field but) as a K-module; secondly the K-module M of all continuous mappings of G into K [where we consider the discrete topology in K]. The principal result is the proof of the fact that these two representations of G are equivalent. The main tool used in the proof is an "approximate" normal basis theorem for L over K.

R. Baer (Urbana, Ill.).

Flanders, Harley. The norm function of an algebraic field extension. II. Pacific J. Math. **5** (1955), 519–528.

Let K be a finite extension of a field k. In a previous paper [same J. **3** (1953), 103–113; MR **14** 1065], the author characterized the norm function $N_{K/k}(\alpha)$ $(\alpha \in K)$, from K to k by some simple properties. For the proof, he used the following lemma: let $\omega_1, \cdots, \omega_n$ be a basis of K/k and $X_1, \cdots, X_n$ corresponding indeterminates. Then $N_{K/k}(\omega_1 X_1+\cdots+\omega_n X_n)$ is a power of an irreducible polynomial of $X_1, \cdots, X_n$ with coefficients in k. In the present paper, the author gives an elementary proof of the above result without using much of field theory. Some applications are also mentioned. *K. Iwasawa.*

Zassenhaus, Hans. Über eine Verallgemeinerung des Henselschen Lemmas. Arch. Math. **5** (1954), 317–325.

The following generalization is given of Hensel's well-known lemma [cf., e.g., van der Waerden, Moderne Algebra, Tl. 1, 3. Aufl., Springer, Berlin, 1950, p. 263; MR **12**, 236]. Let k be a field with a complete valuation φ and let $\mathfrak{o}$ be the ring of integers of k $(a \in \mathfrak{o} \Leftrightarrow \varphi(a) \leqq 1)$ and $\mathfrak{p}$ be the maximal prime ideal of $\mathfrak{o}$ $(a \in \mathfrak{p} \Leftrightarrow \varphi(a) < 1)$. Further let $\mathfrak{D}$ be an associative torsionfree $\mathfrak{o}$-ring (i.e. for $a \in \mathfrak{o}$ and $A \in \mathfrak{D}$ if $aA=0$ then $a=0$ or $A=0$) with unit and finitely generated over $\mathfrak{o}$. Then, if in the hypercomplex system $\mathfrak{D}/\mathfrak{D}\mathfrak{p}$ a system of matrix units e_{ij}^{μ} $(\mu=1, \cdots, r,\ i, j=1, \cdots, f_\mu)$ satisfying

(1) $e_{ij}^{\mu} e_{kl}^{\nu}=0$ if $\mu \neq \nu$ or $j \neq k$, $=e_{il}^{\mu}$ if $\mu=\nu$ and $j=k$

is given there will be in $\mathfrak{D}$ a system E_{ij}^{μ} satisfying

(2) $E_{ij}^{\mu} E_{kl}^{\nu}=0$ if $\mu \neq \nu$ or $j \neq k$, $=E_{il}^{\mu}$ if $\mu=\nu$ and $j=k$,

E_{ii}^{μ} belonging to the residue class e_{ij}^{μ}. The case $r=2$, $f_1=f_2=1$ is equivalent to Hensel's lemma. First the author proves the somewhat analogous purely algebraic theorem A: If in a ring $\mathfrak{S}$ elements $e_{ij}^{\mu}(\mu=1, \cdots, r;\ i, j=1, \cdots, f_\mu)$ are given which satisfy

(3) $e_{ij}^{\mu} e_{kl}^{\nu}=0(\mathfrak{N})$ if $\mu \neq \nu$ or $j \neq k$, $=e_{il}^{\mu}(\mathfrak{N})$ if $\mu=\nu$ and $j=k$,

where $\mathfrak{N}$ is a two-sided nilideal of $\mathfrak{S}$, then there exist elements E_{ij}^{μ} satisfying (2) and $e_{ij}^{\mu}=E_{ij}^{\mu}(\mathfrak{N})$. The case $r=1=f_1$ is ascribed to Brauer. The proof is somewhat lengthy and uses mathematical induction. With the aid of an $\mathfrak{o}$ basis $A_1, \cdots, A_n$ of $\mathfrak{D}$ an extension of the valuation of φ to a complete pseudo-valuation $\varphi_{\mathfrak{D}}$ of $\mathfrak{D}$ can be given by putting $\varphi_{\mathfrak{D}}(\sum_{i=1}^{n} A_i A_i)=\max_i \varphi(A_i)$ $(A_i \in \mathfrak{o}$ and $A=\sum A_i A_i \in \mathfrak{D})$ clearly satisfying $\varphi_{\mathfrak{D}}(A)=\varphi(A)$ if

$$A \in \mathfrak{o},\ \varphi_{\mathfrak{D}}(A+B) \leqq \max(\varphi_{\mathfrak{D}}(A), \varphi_{\mathfrak{D}}(B)),$$

$$\varphi_{\mathfrak{D}}(A \cdot B) \leqq \varphi_{\mathfrak{D}}(B)\ (A, B \in \mathfrak{D}).$$

The set of all $A \in \mathfrak{D}$ with $\varphi_{\mathfrak{D}}(A) \leqq \eta \leqq 1$ is a two-sided ideal denoted by $\mathfrak{D}(\eta)$. Now arbitrary elements E_{ij}^{μ} from the residue class e_{ij}^{μ} satisfy (3) with $\mathfrak{N}=\mathfrak{D}(\varkappa)/\mathfrak{D}(\varkappa^2)$, for some $\varkappa<1$, which is a nilideal in $\mathfrak{S}=\mathfrak{D}/\mathfrak{D}(\varkappa^2)$. Repeated application of the theorem A gives systems of matrix units ${}_{\tau}E_{ij}^{\mu}$ which converge to the required system if τ tends to infinity. After defining equivalence of two systems $e_{ij}^{\alpha\mu}$ $(\alpha=\pm 1)$ both satisfying (1), it is derived that equivalence mod the two-sided nilideal $\mathfrak{N}$, respectively mod $\mathfrak{p}\mathfrak{D}$, implies equivalence in $\mathfrak{S}$, respectively $\mathfrak{D}$, from which it follows: Every semi-primary ring $\mathfrak{S}$ with unit admits a Remak decomposition of the left $\mathfrak{S}$-module $\mathfrak{S}$. Every two such decompositions are operator isomorphic (by an inner automorphism), and the similar theorem for the $\mathfrak{D}$ left module $\mathfrak{D}$. Also some remarks about the relation with the modular representations of finite groups are given. The paper concludes with considerations about the connection of the representations of $\mathfrak{D}$ over $\mathfrak{o}$ and of $\mathfrak{D}/\mathfrak{p}\mathfrak{D}$ over $\mathfrak{o}/\mathfrak{p}$. Since the concepts used are rather involved we shall not formulate the result here.

J. Verhoeff (Zbl **56**, 266).

Ribenboim, P. Sur une conjecture de Krull en théorie des valuations. Nagoya Math. J. **9** (1955), 87–97.

Let A be a completely integrally closed primary integral domain. Nagata [same J. **4** (1952), 29–33; MR **13**, 904] has exhibited such a domain which is not a valuation ring, thus providing a counterexample to a conjecture of Krull. In the present paper the author seeks supplementary conditions sufficient to ensure that A be a valuation ring. These conditions depend on the structure of the unique prime ideal $\mathfrak{p}$ in A. A known condition is that $\mathfrak{p}$ be principal. More generally, suppose that $\mathfrak{p}$ is the union of an increasing sequence of principal ideals At_r $(r=1, 2, \cdots)$. Let

$$\mu_r(a)=\sup\left\{\frac{m}{n} \,\middle|\, Aa^n \subseteq At_r^m\right\},\ \nu_r(a)=\inf\left\{\frac{m}{n} \,\middle|\, At_r^m \leqq Aa^n\right\},$$

where m and n are integers. It is shown that if

(*) $$\lim_{r \to \infty} \mu_r(a)/\nu_r(a)=1$$

for each non-zero $a \in \mathfrak{p}$, then A is a valuation ring. (When $\mu_r(a)=\nu_r(a)=s_r$, one can write $Aa=(At_r)^{s_r}$ [cf. also Samuel, Ann. of Math. (2) **56** (1952), 11–21; MR **14**, 128], so that condition (*) is a requirement that each principal ideal in A be "asymptotically" a power of the ideals At_r. Conditions are also given in topological terms. The conditions given are necessary as well as sufficient.

B. N. Moyls (Vancouver, B.C.).

Seidenberg, A. On homogeneous linear differential equations with arbitrary constant coefficients. Pacific J. Math. **5** (1955), 599–606.

Let K be a field, (u_n), $0 \leqq n < \infty$, be an infinite sequence of indeterminates, (c_i), $0 \leqq i \leqq m$, be a finite sequence of additional indeterminates, $l_n=\sum_{0 \leqq i \leqq m} c_i u_{i+n}$, $R_n=K[u_0, \cdots, u_n]$, $S_n=K(c_0, \cdots, c_m)[u_0, \cdots, u_n]$, p be the

ideal $(l_0, \cdots, l_t)$ of S_{t+m}, $P=\mathfrak{p}\cap R_{t+m}$, A be the ideal of R_{t+n} generated by the $(m+1)$-rowed minors of the matrix (u_{i+j}), $0\leq i\leq m$, $0\leq j\leq t$, where $t\geq m$. It is shown that $A=P$. If in the above K is an ordinary differential field, u_n represents the nth derivative of the indeterminate u, $c_0, \cdots, c_m$ are constants, and t is allowed to become infinite, this shows that the differential ideal $[l_0]\cap K\{u\}$ (where $[l_0]$ is the differential ideal generated by l_0 in the differential ring $K(c_0, \cdots, c_m)\{u\}$) is the same as the ideal of $K\{u\}$ generated by all the $(m+1)$-rowed minors of the matrix (u_{ij}), $0\leq i\leq m$, $0\leq j<\infty$.

E. R. Kolchin (New York, N.Y.).

Kolchin, E. R. On the Galois theory of differential fields. Amer. J. Math. **77** (1955), 868–894.

L'A. reprend sa théorie de Galois des extensions fortement normales [même J. **75** (1953), 753–824; MR **15**, 394, 1140] en exploitant le fait que les groupes en question sont dotés d'une structure analogue à celle des variétés de groupes [Weil, Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948; MR **10**, 621], la définition des variétés de groupes étant ici élargie de manière à englober des variétés non irréductibles.

Soit $\mathcal{T}$ un corps différentiel partiel de caractéristique nulle et de corps des constantes $\mathcal{C}$ algébriquement clos. Soit $\mathcal{T}^*$ une extension universelle de $\mathcal{T}$ de corps des constantes $\mathcal{C}^*$. Soit $\mathcal{G}$ une extension fortement normale de $\mathcal{T}$. L'auteur appelle groupe de Galois de $\mathcal{G}/\mathcal{T}$ le groupe des automorphismes de $\mathcal{G}\langle c^*\rangle/\mathcal{T}\langle c^*\rangle$, qui est canoniquement identifiable à l'ensemble des isomorphismes de $\mathcal{G}/\mathcal{T}$. Pour tout isomorphisme σ de $\mathcal{G}/\mathcal{T}$, soit $\mathcal{C}(\sigma)$ le corps des constantes de $\mathcal{G}\langle\sigma\mathcal{G}\rangle$.

Une application f d'une variété de groupe G (le domaine universel étant $\mathcal{C}^*$), dans une variété de groupe H est un homomorphisme rationnel si et seulement si f est un homomorphisme de groupes tel que (i) pour tout $s\in G$ on a $\mathcal{C}(f(s))\subset\mathcal{C}(s)$ et (ii) si s' est une spécialisation (sur $\mathcal{C}$) de s, alors $f(s')$ est une spécialisation de $f(s)$. La notion d'homomorphisme rationnel s'étend ainsi à une classe de groupes comprenant les variétés de groupes et les groupes de Galois d'extension fortement normales, ce qui permet à l'A. d'établir que le groupe de Galois de toute extension fortement normale est birationnellement isomorphe à une variété de groupe. [Ainsi que le signale l'A., ce théorème avait déjà été démontré par Matsumura dans le cas des groupes irréductibles, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28** (1954), 283–292; MR **16**, 563.] Kolchin donne, de plus, une réciproque partielle à ce théorème: Toute variété de groupe irréductible est birationnellement isomorphe au groupe de Galois d'une extension fortement normale.

Dans la dernière partie du mémoire, l'A. utilise des notions cohomologiques. Les 1-cocycles du groupe de galois $\mathfrak{G}$ de $\mathcal{G}/\mathcal{T}$ dans la variété de groupe G (de domaine universel $\mathcal{C}^*$) sont les applications f de $\mathfrak{G}$ dans la variété $G^\dagger$ de domaine universal $\mathcal{T}^*$ définie par G, telles que (i) $\mathcal{G}(f(\sigma))\subset\mathcal{G}\langle\sigma\mathcal{G}\rangle$ pour tout $\sigma\in\mathfrak{G}$; (ii) si $\sigma'\in\mathfrak{G}$ est une spécialisation de σ, alors $f(\sigma')$ est une spécialisation de $f(\sigma)$; (iii) pour tout couple σ, $\sigma'\in\mathfrak{G}$, on a $f(\sigma\cdot\sigma')=f(\sigma)\cdot\sigma f(\sigma')$. Deux 1-cocycles f, g seront dits cohomologues s'il existe un point $\alpha\in G^\dagger$, rationnel sur $\mathcal{G}$ et tel que pour tout $\sigma\in\mathfrak{G}$, on ait $g(\sigma)=\alpha^{-1}f(\sigma)\sigma(\alpha)$, ce qui permet de définir l'ensemble de cohomologie $H^1(\mathfrak{G}, G)$. L'A. montre que $H^1(\mathfrak{G}, G)=1$ lorsque $G=\mathrm{GL}(n)$ ou que $\mathcal{T}$ est algébriquement clos. Si G est commutatif, $H^1(\mathfrak{G}, G)$ est un groupe dont tous les éléments sont d'ordre fini.

L'A. déduit enfin de sa théorie qu'une extension fortement normale $\mathcal{G}$ de $\mathcal{T}$ est de Picard-Vessiot si et seulement si le groupe de Galois de $\mathcal{G}/\mathcal{T}$ est birationnellement isomorphe à un groupe algébrique de matrices.

G. Papy (Bruxelles).

See also: Tornheim, p. 463 ;Havel, p. 522; Klingenberg, p. 522.

Groups, Generalized Groups

Büchi, J. Richard, and Wright, Jesse B. The theory of proportionality as an abstraction of group theory. Math. Ann. **130** (1955), 102–108.

The algebraic content of the paper is as follows: A quaternary relation R on a set A is called a regular bi-equivalence if it satisfies eight postulates of which we mention only: (8) $(Eu)R(x, y, u, v)$. A regular bi-equivalence R and a preassigned element e define a group operation $(\cdot)$ on A by the requirement that $R(x, e, x\cdot y, y)$. Then R is the proportionality relation on $(A, \cdot)$ in the sense that $R(x, y, u, v)$ means $xy^{-1}=uv^{-1}$. Conversely, every proportionality relation is a regular bi-equivalence. The permutations of A which preserve a regular bi-equivalence R form a group which coincides with the holomorph of each group defined (as above) by R. The rest of the paper consists of a number of remarks (and thirteen "notes") which indicate that the authors are primarily interested in metamathematics. To quote out of context: "All significant ideas connected with subgroups and in particular normal subgroups can without difficulty be reformulated in the theory of regular bi-equivalences." (Note 11, third sentence.) *R. H. Bruck* (Madison, Wis.).

Tsuboi, Teruo. Note on double coset decomposition. I. Sci. Rep. Saitama Univ. Ser. A. **2** (1955), 1–5.

Let H, K be subgroups of a finite group G, and let $S(H, K)$ be the intersection for all x in G of the subgroups $(x^{-1}Hx)K\cap K(x^{-1}Hx)$. The double cosets HaK form a group under complex multiplication if and only if $S(H, K)$ is a normal subgroup of G, in which case HK is a normal subgroup of G and the double cosets coincide with the cosets of G modulo HK. *D. G. Higman.*

Goheen, H. E. On a theorem of Zassenhaus. Proc. Amer. Math. Soc. **5** (1954), 799–800.

In the proof of his theorem [Proc. Glasgow Math. Assoc. **1** (1952), 53–63; MR **14**, 614] that if, in a finite group G, the normaliser of every Abelian subgroup is also its centraliser then G is Abelian, H. J. Zassenhaus has considered separately the possibilities that the centre of G is, or is not, the identity subgroup of G. In this paper the author gives an alternative proof for the first of these possibilities. *M. C. R. Butler* (Zbl **56**, 255).

Nielsen, Jakob. A basis for subgroups of free groups. Math. Scand. **3**, (1955) 31–43.

The author gives a new proof of the fact that every subgroup of a free group is free, and points out that his method can sometimes be used to determine whether a given element of a free group belongs to the subgroup generated by a given set. *B. Jónsson* (Berkeley, Calif.).

Szép, J. Zur Theorie der faktorisierbaren Gruppen. Acta Sci. Math. Szeged **16** (1955), 54–57.

The author proves the following two theorems: 1. A factorizable group $G=AB$ is not simple if A is abelian,

B has a non-trivial center and the order of B does not exceed the order of A. 2. If a factorizable group $G=AB$ with $A\cap B=1$ has a non-trivial center consisting of the elements a_ib_i, $a_i\in A$, $b_i\in B$, then the distinct elements a_i and b_i form groups A' and B', and the group $G'=A'B'$ generated by them is abelian. It follows that in a finite factorizable group $G=AB$ with $A\cap B=1$ for which A and B have trivial centers the square of the order of the center of G does not exceed the order of G. *K. A. Hirsch.*

Šik, František. Die Anwendung der Polarität auf die direkten Produktzerlegungen einer Gruppe. Czechoslovak Math. J. **5** (**80**) (1955), 61–75. (Russian summary)

The author defines a special type of polarity on the subsets of a set G in terms of a reflexive, transitive binary relation $\leqq$ with a least element e and a symmetric, anti-reflexive binary relation D satisfying (a) $x\geqq y$, $xDy\Rightarrow e\geqq y$; (b) eDe; (c) xDy, $z\leqq x\Rightarrow zDy$; (d) $xDy\Rightarrow$ there exists a $z\in G$ such that $z\not\leqq e$, $z\leqq x$, $z\leqq y$. The resulting theory is applied to obtain structure theorems of finite cyclic groups, of groups possessing a unique direct product decomposition and a theorem analogous to that of Schmidt-Remak for the direct product decomposition of a group satisfying the minimal property for normal subgroups. *L. J. Paige* (Los Angeles, Calif.).

McLain, D. H. On locally nilpotent groups. Proc. Cambridge Philos. Soc. **52** (1956), 5–11.

The following results are proved (among others): (1) If the Frattini subgroups $\phi(S)$ of every finitely generated subgroup S of G contains the commutator subgroup S' of S, then $G'\leqq\phi(G)$. (2) If $[N, S]<N$ for every normal subgroup $N\neq 1$ of every finitely generated subgroup S of G, then every minimal normal subgroup of G is part of the center of G. (3) If every finitely generated subgroup of G is of finite class, then the minimum (maximum) condition for normal subgroups of G implies the minimum (maximum) condition for all subgroups of G. *R. Baer.*

Suprunenko, D. On nilpotent matrix groups. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **1953**, no. 15, 3–6. (Russian)

Making use of his earlier paper [same Uč. Zap. **1951**, no. 12, 74], the author develops A. I. Mal'cev's results concerning soluble matrix groups [Mat. Sb. N.S. **28**(**70**) (1951), 567–588; MR **13**, 203]. He proves the theorem: the center of an arbitrary irreducible nilpotent subgroup Γ of the complete linear group $GL(n)$ over an arbitrary field has a finite index in Γ.

S. N. Černikov (RŽMat **1953**, no. 82).

Suprunenko, D. A. On a property of nilpotent matrix groups. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 273–274. (Russian)

The author proves that the general linear group $GL(n, F)$ over an algebraically closed field F contains only a finite number of non-conjugate maximal nilpotent subgroups of a given class of nilpotency. If the field F is arbitrary and G a finite nilpotent subgroup of $GL(n, F)$ of class l and the characteristic of F is not divisible by l, then the index of the center of G does not exceed a certain number $N(n, l)$ depending on n and l only. This last theorem again makes use of results contained in the author's paper reviewed above.

K. A. Hirsch (London).

Suprunenko, D. A. Locally nilpotent irreducible subgroups of the full linear group. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 41–44. (Russian)

The author has proved recently [same Dokl. (N.S.) **99** (1954), 23–25; MR **16**, 793] that all maximal transitive nilpotent subgroups of the symmetric groups are conjugate. The present note establishes an analogous result for the general linear group $GL(n, F)$ over an algebraically closed field F. All maximal locally nilpotent irreducible subgroups of $GL(n, F)$ are conjugate, and every locally-nilpotent irreducible subgroup of $GL(n, F)$ is contained in a maximal locally nilpotent subgroup of $GL(n, F)$. The proofs are partly based on results of the author's paper reviewed second above. An interesting lemma states that if Γ is an irreducible nilpotent subgroup of $GL(n, F)$ and the center of Γ is the multiplicative group of scalar matrices, then the intersection $G=\Gamma\cap SL(n, F)$, where $SL(n, F)$ is the special linear group, is finite, and the order of G contains only prime divisors of n.

K. A. Hirsch (London).

★**Kaloujnine, Leo. Über gewisse Beziehungen zwischen einer Gruppe und ihren Automorphismen.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 164–172. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

Let G be a group with elements $a, b, c, \cdots$; A a group of automorphisms of G with elements $\alpha, \beta, \gamma, \cdots$ written as exponents; $K(G; A)=\{a^{-1}a\alpha\}$, $a\in G$, $\alpha\in A$, so that $K(G; A)$ is a generalization of the commutator group and is normal in G; $Z(G; A)$ the set of fixed elements g of G, i.e. $g^{-1}g^{\alpha}=1$. For two groups of automorphisms A and B any two of the following three conditions imply the third: I. $K(G; A)\subseteqq Z(G; B)$; II. $K(G; B)\subseteqq Z(G; A)$; III. $[A, B]=1$. Let H be a subgroup of H and collect all the automorphisms α for which $a^{-1}a^{\alpha}\in H$ for all $a\in G$ and $h^{-1}h^{\alpha}=1$ for all $h\in H$. They form a group A of automorphisms which the author calls the "stability" group of the chain $G\supseteqq H\supseteqq 1$; $A=\mathrm{Stab}(G\supseteqq H\supseteqq 1)$ can also be characterized as the largest group of automorphisms for which $K(G; A)\subseteqq H\subseteqq Z(G; A)$. A is, of course, abelian and if we set $R=K(G; A)$, then R is abelian. Generalizing these facts the author takes a chain of subgroups $G=H_0\supseteqq H_1\supseteqq\cdots\supseteqq H_m=1$ and the collection of automorphisms α which leave for each $i=1, 2, \cdots, m$ the (right or left) cosets of H_{i-1} in H_i elementwise invariant. They form a group $\mathrm{Stab}(H_0\supseteqq H_1\supseteqq\cdots\supseteqq H_m)$. Conversely, if C is a group of automorphisms which for a suitable chain satisfies $C\subseteqq\mathrm{Stab}(H_0\supseteqq H_1\supseteqq\cdots\supseteqq H_m)$ then this chain is called a C-central series. Special cases are the C-commutator groups $K_0(G; C)=G$, $K_{i+1}(G; C)=K(K_i(G; C))$. A group is called C-nilpotent of class c if $K_{c-1}(G; C)\neq 1$, $K_c(G; C)=1$. If in the above chain the H_i are normal subgroups of G, then the group $\mathrm{Stab}(H_0\supseteqq H_1\supseteqq\cdots\supseteqq H_m)$ is nilpotent (in the usual sense) of a class not exceeding $m-1$. Without the assumption on the normality of the H_i the stability group is soluble in at most $m-1$ steps. If B is a group of automorphisms for which G is B-nilpotent and if the descending B-central series with the terms $K_i(G; B)$ consists of normal subgroups of G, then the group $K_1(G; B)$ is nilpotent (in the usual sense) and the terms $K_i(G; B)$ form a central series of $K_1(G; B)$.

Some related results can be found in a paper by F. Haimo [Trans. Amer. Math. Soc. **78** (1955), 150–167; MR **16**, 794].

K. A. Hirsch (London).

Maurer, I. Les groupes de permutations infinies. Gaz. Mat. Fiz. Ser. A. **7** (1955), 400–408. (Romanian. Russian and French summaries)

After systematically deriving the principal known results concerning the normal subgroups, normal series and automorphisms of the infinite symmetric group S_∞, and its generalizations $S_\infty(A)$ of Specht type, the author introduces a topology into S_∞ and states some properties of S_∞ as a topological group, e.g., its simplicity, the proofs of which are to appear in another paper. The topology may be described as follows: To each finite sequence $n_1, \cdots, n_k$ of positive integers there corresponds a fundamental neighborhood of the identity consisting of all those $\pi \in S_\infty$ such that $\pi n_i = n_i$ $(i=1, \cdots, k)$.

D. G. Higman (Missoula, Mont.).

Artin, Emil. The orders of the classical simple groups. Comm. Pure Appl. Math. **8** (1955), 455–472.

Complétant un résultat antérieur [mêmes Comm. **8** (1955), 355–365; MR **17**, 12], l'auteur prouve que, parmi les groupes simples finis connus (i.e. les groupes alternés, les groupes classiques, les groupes de type exceptionnel récemment découverts par Chevalley, et les 5 groupes de Mathieu), il n'y a pas de coïncidences entre les ordres de ces groupes sauf dans les cas connus depuis Dickson [Linear groups, Teubner, Leipzig, 1901]. Il arrive à réduire considérablement le nombre de vérifications nécessaires par la méthode suivante: 1) pour la comparaison entre les groupes alternés et les groupes du type „de Lie", sur un corps de caractéristique p, on remarque que si N est l'ordre d'un groupe de ce dernier type, M la plus haute puissance de p divisant N („p-contribution" à N), on a $N<M^3$, et d'autre part, on évalue aisément la p-contribution à une factorielle; 2) sauf dans un petit nombre de cas bien préeisés et faciles à traiter séparément, p est le nombre premier dont la „contribution" à l'ordre N est la plus grande, ce qui dispose facilement de la comparaison entre deux groupes sur des corps de caractéristique différente; 3) pour le cas le plus difficile de corps de même caractéristique, l'auteur considère les différents polynômes cyclotomiques ϕ_n tels que $\phi_n(p)$ divise n; il prouve d'abord que si α et β sont les deux plus grandes valeurs de s, la connaissance de N, p, α et β détermine le groupe sans ambigüité; en outre, si $p>2$, α et β sont les deux plus grands exposants de p modulo les nombres premiers $\neq p$ divisant N, et par suite α et β sont déterminés par N et p; pour $p=2$ enfin, on applique la même idée mais il y a un nombre fini d'exceptions à examiner séparément.

J. Dieudonné.

Chevalley, C. Sur certains groupes simples. Tôhoku Math. J. (2) **7** (1955), 14–66.

Most of the known abstract simple groups have been obtained by starting with a simple Lie group G (over the complex numbers), observing that it is locally isomorphic to an algebraic matrix group Γ and then considering this algebraic matrix group over other constant fields K. By this method L. Dickson [Linear groups, Teubner, Leipzig, 1901] has obtained finite simple groups by taking for K any finite field. Recently Dieudonné [Sur les groupes classiques, Hermann, Paris 1948; MR **9**, 494] has obtained simple groups by defining the group geometrically in a manner which works for an arbitrary constant field K. The methods of Dickson and Dieudonné make it necessary to study each type of simple Lie groups separately. This special analysis has been carried out in the case of the classical groups and of one of the exceptional groups G_2 [Dickson, Trans. Amer. Math. Soc. **2** (1901), 363–394; Math. Ann. **60** (1905), 137–150]. The author states that he has applied analogous methods to the exceptional groups F_4, E_6 and E_7 in an as yet unpublished article.

In the present paper a method is developed which uses general structure theorems of semi-simple Lie algebras and groups, in such a manner that the transfer to an arbitrary constant field K can be carried out simultaneously and uniformly for all complex simple Lie groups. One of the main steps used in this procedure is an important improvement of the known behavior of the bracket of root-vectors in a complex semi-simple Lie algebra g. Let $\mathfrak{h}$ be a Cartan subalgebra of g, and let r vary over the roots with respect to $\mathfrak{h}$. Theorem 1 asserts that elements X_r can be chosen such that $[X_r, X_{-r}]=H_r$ and (if r, s and $r+s$ are roots) $[X_r, X_s]=N_{r,s}X_{r+s}$ where the $N_{r,s}$ are rational integers; in fact, $N_{r,s}=\pm(p+1)$ where p is the largest integer $m\geq 0$ such that $s-mr$ is a root. The author's proof of Theorem 1 consists in reducing the problem to Lie algebras of rank 2 and examining each type of complex simple Lie algebra of rank 2 separately. Another proof of this theorem (not yet published) has been given by Harish-Chandra.

Having shown that every complex simple Lie algebra g has a basis $H_1, H_2, \cdots, X_r, X_s, \cdots$ where the structure constants are integers, it follows that one can use this basis to construct a Lie algebra g_K over an arbitrary constant field K. This is carried out in § III. The next step consists in constructing a group G corresponding to this Lie algebra g_K. For this purpose let χ be a homorphism of the additive group P generated by the roots into the multiplicative group K^* of the field K. Denote by $h(\chi)$ the automorphism of the vector space g_K which leaves the elements of $\mathfrak{h}_K$ fixed and transforms X_r into $\chi(r)X_r$. The mapping $\chi \to h(\chi)$ is an isomorphism of the group of homomorphisms of P into K^* with a group $\mathfrak{H}$ of automorphisms of g_K. Next a one-parameter group of linear transformations $x_r(t)$ of the vector space g_K is constructed for each root r where t varies over K. It is shown that $h(\chi)x_r(t)h(\chi)^{-1}=x_r(\chi(r)t)$ and that there exists an homomorphism Φ_r of $\mathrm{SL}(2, K)$ into the group of automorphisms of g_K such that

$$\Phi_r\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}=x_{-r}(t),\ \Phi_r\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}=x_r(t),\ \Phi_r\begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix}=h(\chi_r)$$

where χ_r is the homomorphism of P into K^* given by $\chi_r(u)=z^{u(H_r)}$ for every $u \in P$. Let G be the group generated by $\mathfrak{H}$ and the $x_r(t)$. The elements of G are thus automorphisms of the vector space g_K. It is verified now that for a given field K the structure of G depends only on the original complex simple Lie algebra g.

One has now to study the structure of the group G. This is done by introducing the subgroups $\mathfrak{U}$ and $\mathfrak{B}$ corresponding to the positive and negative roots respectively. In this way results are obtained some of which are analogous to well-known properties of complex semi-simple Lie groups. However, if K is a finite field of characteristic p with q elements, $\mathfrak{U}$ and $\mathfrak{B}$ are groups of order q^N (where N is the number of positive roots). The order of the group G can be determined explicitly; it is

$$(q-1)^l q^N \sum_{w \in W} q^{N(w)},$$

where l is the rank of g, W the Weyl group and $N(w)$ the number of roots $r>0$ for which $w(r)<0$. It follows that $\mathfrak{U}$ and $\mathfrak{B}$ are Sylow subgroups of G. Various other results on the structure of G and certain of its subgroups are

obtained; for example if p' is a prime number not equal to the characteristic p of K and if p' does not divide the order of the Weyl group then the p'-Sylow subgroups of G are commutative.

§ IV is to a large extent devoted to the study of a normal subgroup of G. Let G' be the subgroup of G generated by $\mathfrak{U}$ and $\mathfrak{B}$. It is shown that G' is a normal subgroup of G and that G/G' is isomorphic to a (explicitly described) factor group of $\mathfrak{H}$. A detailed analysis of the structure of $G' \cap G$ follows which leads to the following result: G' is the commutator subgroup of G and G' is a simple group with the following exceptions: a) K has two elements and the complex Lie algebra g is of type A_1, B_2 or G_2; b) K has three elements and g is of type A_1. The order of the group G' is computed explicitly in terms of explicitly known invariants of g (such as the rank and Poincaré polynomial) and the number q of elements of K. Now the orders of the known simple finite groups have recently been determined by E. Artin in the paper reviewed above. It follows that the above finite simple groups corresponding to some of the exceptional simple Lie groups are new. The paper closes with the statement of a number of unsolved problems. *F. I. Mautner.*

Reiner, Irving. Automorphisms of the symplectic modular group. Trans. Amer. Math. Soc. **80** (1955), 35–50.

Let Γ_{2n} be the symplectic modular group of order $2n$, i.e. the set of $2n$-rowed matrices $\mathfrak{T}$ with integral coefficients satisfying $\mathfrak{J}\mathfrak{B}\mathfrak{T}'=\mathfrak{J}$ where $\mathfrak{J}=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and $I=I^{(n)}$, $0=0^{(n)}$ are the n-rowed identity and zero matrix respectively. Let $\mathfrak{A}$ be a matrix satisfying $\mathfrak{A}\mathfrak{J}\mathfrak{A}'=\pm\mathfrak{J}$. Evidently $\mathfrak{T}\to\mathfrak{A}\mathfrak{T}\mathfrak{A}^{-1}$ is an automorphism of Γ_{2n}. The author proves that for $n>2$, there is no other automorphism. For $n=2$, the author adds another type of automorphism $\mathfrak{T}\to\chi(\mathfrak{T})\mathfrak{A}\mathfrak{T}\mathfrak{A}^{-1}$, where $\chi(\mathfrak{T})$ is a homomorphism of Γ_{2n} into ± 1. That is, let Γ_4^* be the group generated by the square elements of Γ_4. The factor group Γ_4/Γ_4^* gives the whole nature of $\chi(\mathfrak{T})$. *L. K. Hua.*

Koecher, Max. Einheiten schiefsymmetrischer Matrizen. Math. Nachr. **13** (1955), 367–382.

The author studies the symplectic unimodular group U_{2n}, i.e. the group consisting of symplectic matrices of order $2n$ with integral elements. He considers some subgroups of that group, shows that, taken together, they generate the whole group, and that each of them has a finite system of generators, which he determines explicitly, along with some relations between these generators. Finally, he studies the automorphisms of the group U_{2n} which can be extended to automorphisms of the real symplectic group $Sp_{2n}(R)$; but his results have now been completely superseded by those of I. Reiner [see the paper reviewed above] who has succeeded in determining all automorphisms of U_{2n}. *J. Dieudonné.*

Conrad, Paul. Extensions of ordered groups. Proc. Amer. Math. Soc. **6** (1955), 516–528.

Let G be a linearly ordered group ("o-group"). Let Γ be the set of all pairs of convex subgroups (G^γ, G_γ) of G such that G^γ covers G_γ. Γ is linearly ordered by inclusion, and the rank of G is defined to be the order type of Γ. Each G_γ is normal in G^γ and the archimedean o-group G^γ/G_γ is called a component of G. G is said to be d-closed if $nG=G$ for every positive integer n. G is called an a-extension of a subgroup H if there is an element of H in every archimedean class $G^\gamma-G_\gamma$ of G. G is said to be a-closed if it does not admit a proper a-extension. In a previous paper [same Proc. **5** (1954), 323–328; MR **15**, 849] the author showed that every o-group admits an a-closed a-extension.

Consider the following proposition $P(G)$ concerning an o-group G: "G admits an a-closed a-extension each component of which is o-isomorphic with the additive group R of real numbers". $P(G)$ is true if G is abelian, by Hahn's Theorem. The author gives an example of an o-group G for which $P(G)$ is false. However, one of the main results of the paper is that $P(G)$ is true if each component of G is d-closed. All the logical relations between the properties of a-closure and d-closure of G and of all the components of G are elucidated, with one exception: if G is a-closed and each component of G is $\cong R$, is G itself d-closed?

Under the assumption that the rank Γ of G is well-ordered, the answer to this question is in the affirmative. In fact, a necessary and sufficient condition for d-closure of G is given (with well ordered (w.o.)Γ) from which the following corollaries are deduced: (I) if G is abelian then G is d-closed if and only if each component of G is d-closed; (II) if each component of G is $\cong R$ then G is d-closed. An example is given to show that both I and II can be false if Γ is not w.o. It is shown that a-closure implies d-closure if Γ is the ordinal 2, but the question is left open for arbitrary w.o. Γ. *A. H. Clifford.*

Frink, Orrin. Symmetric and self-distributive systems. Amer. Math. Monthly **62** (1955), 697–707.

A binary operation $a+b$ is said to be symmetric if it obeys the law $(a+b)+(c+d)=(a+c)+(b+d)$. Such an operation is in general neither commutative nor associative. The author derives the basic algebraic properties of systems with a single symmetric operation. The endomorphisms of such a system constitute a "pseudo-ring" that is, a system with a symmetric addition and an associative multiplication which is distributive with respect to addition. Insofar as they apply to quasigroups, many of the results have been obtained previously by K. Toyoda [Tôhoku Math. J. **46** (1940), 239–251; Proc. Imp. Acad. Tokyo **16** (1940), 161–164; MR **2**, 6] and by the reviewer [Trans. Amer. Math. Soc. **49** (1941), 392–409; MR **2**, 218]. *D. C. Murdoch* (Vancouver, B.C.).

Sagastume Berra, Alberto E. Divisibility in groupoids. I, II. Univ. Nac. Eva Peron. Publ. Fac. Ci. Fisicomat. Serie Segunda. Rev. **5** (1954), 67–122. (Spanish)

The author defines a groupoid $\mathcal{G}$ to be a system closed under a single-valued associative and commutative binary operation (multiplication) and possessing an identity element. [Current English (French) usage: $\mathcal{G}$ is a commutative multiplicative semigroup (demigroup) with identity element.] The contribution of the paper is the study of an abstract notion of divisibility. The paper is in two parts, which may be discussed together as follows: Let $\mathcal{S}$ be the set of all transitive binary relations on $\mathcal{G}$: $a|b\ (D)$ means a is in the relation D to b. Among the elements of $\mathcal{S}$ are A: $a|b\ (A)$ if and only if $b=ac$ for some c; and T: $a|b\ (A)$ if and only if $b=ac$ for some c; and T: $a|b\ (T)$ for all a, b. Turn $\mathcal{S}$ into a lattice by defining $U\subseteq V$, for elements U, V of $\mathcal{S}$, if $a|b\ (V)$ implies $a|b\ (U)$. An element D of $\mathcal{S}$ is a divisibility of $\mathcal{G}$ provided $D\subseteq A$. The divisibilities of $\mathcal{G}$ form a complete sublattice $\mathcal{L}$ of $\mathcal{S}$. If R is a fixed divisibility, an element D of $\mathcal{S}$ is a divisibility relative to R provided $D\subseteq R$; then clearly D is a divisibility. The divisibilities relative to a fixed divisibility R form a sublattice $\mathcal{L}(R)$ of $\mathcal{L}=\mathcal{L}(A)$. These are

perhaps the main results of the paper. In Part I the author notes that A is the binary relation corresponding to the usual concept of divisibility. In terms of an arbitrary divisibility D, the concepts of divisor, multiple, unit, associate, ideal, co-ideal have a natural generalization. We say that a is a D-divisor of b, or b is a D-multiple of a, if $a|b$ (D). Then a D-ideal (D-co-ideal) of $\mathcal{G}$ is a non-empty subset of $\mathcal{G}$ which contains the D-multiples (D-divisors) of each of its elements. Similarly for other concepts. As the author shows in considerable detail, the generalized theory closely parallels the usual one. In Part II he pays special attention to divisibilities which are quasi-multiplicative (in the sense that a pair of D-associates a, b and a pair of D-associates c, d lead to a pair of D-associates ac, bd) or are multiplicative (in the sense that $a|b$ (D) implies $ac|bc$ (D) for all c.) The paper contains many examples and special results, including a study of greatest common divisor and least common multiple. *R. H. Bruck.*

Sonneborn, Lee M. On the arithmetic structure of a class of commutative semi-groups. Amer. J. Math. **77** (1955), 783–790.

Let S be a commutative multiplicative semigroup and call two elements of S associated if each divides the other. Guided by a Diophantine problem, the author imposes the following conditions: (1) Every ascending or descending chain of non-associated divisors of any element of S is of finite length. (2) $ab=cd$ if and only if there exist x, y, z, w in S such that $a=xy$, $b=zw$, $c=xz$, $d=yw$. It turns out eventually that the equivalence classes of associated elements of S form a semigroup S' which satisfies (1) (with "non-associated" deleted) (2) and also (3): For any a in S' there is an idempotent e in S' such that $ea=a$. The idempotents of S' form a distributive lattice and S' is a subdirect product of this with a Gaussian semigroup. An example is given to show that "subdirect" cannot be replaced by "direct". *R. H. Bruck.*

Thierrin, Gabriel. Sur une propriété caractéristique des demi-groupes inversés et rectangulaires. C. R. Acad. Sci. Paris **241** (1955), 1192–1194.

Continuing his study of inversed and rectangular semigroups [Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955) 83–92; MR **17**, 10], the author shows that a semigroup D is inversed and rectangular if and only if its right regular partial orderings are stationary on the left, and its left regular partial orderings are stationary on the right. (A relation R is stationary on the right if $baRba'$ implies $xaRxa'$ for all x in D.) *A. H. Clifford.*

Hashimoto, Hiroshi. On the structure of semigroups containing minimal left ideals and minimal right ideals. Proc. Japan Acad. **31** (1955), 264–266.

Let S be a semigroup containing minimal left ideals L_i and minimal right ideals R_k, and let D be the kernel of S. Then S is said to satisfy Condition A (B) if, for each x in S, xD (Dx) is a minimal right (left) ideal of S. Let S_k (T_i) be the set of all x in S such that $xD=R_k$ $(Dx=L_i)$. Then the S_k (T_i) are mutually disjoint right (left) ideals of S such that $S_k \supseteq R_k$ $(T_i \supseteq L_i)$. If S satisfies Conditions A and B, then S is the union of all the R_k and of all the L_i, and $S_kT_i \subseteq S_k \cap T_i$. The structure of S, insofar as the decomposition of S into the subsemigroups $S_k \cap T_i$ is concerned, is thus similar to that of D itself.

A. H. Clifford (New Orleans, La.).

See also: Neumann, p. 448; Robinson, p. 449; Séminaire P. Dubreil, p. 451; Jennings, p. 453; Nöbauer, p. 460; Lazard, p. 508; Krafft, p. 509; Artzy, p. 522; Nevanlinna, p. 525; Marchionna Tibiletti, p. 532.

THEORY OF NUMBERS

Mitrinović, Olga. Sur un théorème de Bernoulli. Bull. Soc. Math. Phys. Macédoine **5** (1954), 30–33 (1955). (Serbo-Croatian. French summary)

Grünbaum, Hugo. On divisibility of numbers. Scripta Math. **21** (1955), 204–207.

Let q be an odd integer, not divisible by 5. Let c be the length of the period in the decimal expansion of $1/q$. Writing any integer n in terms of powers of 10^e, say

$$n=\sum_{j=0}^{m} a_j 10^{ej} \text{ with } 0 \leq a_j < 10^e,$$

then q divides n if and only if q divides $\sum a_j$. This is a generalization of the well-known criterion for divisibility by 3. *I. Niven* (Berkeley, Calif.).

Takács, Lajos. Remark to a paper of P. Turán entitled "On a problem in the history of Chinese mathematics." Mat. Lapok **6** (1955), 27–29. (Hungarian. Russian and English summaries)

The following identity

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k} = \binom{n+k}{k}^2$$

occurs without proof in a book of the Chinese mathematician Le-Jen Shoo from 1867. The proof of this identity was given by P. Turán [Mat. Lapok **5** (1954), 1–6; MR **16**, 13] and also Gy. Szekeres using analytical methods. The present note contains an elementary proof of this identity. *Author's summary.*

Surányi, János. Remarks on a problem in the history of Chinese mathematics. Mat. Lapok **6** (1955), 30–35. (Hungarian. Russian and English summaries)

Two proofs are given for the identity

$$\sum_{i=0}^{k} \binom{k}{i}\binom{l}{i}\binom{n+k+l-i}{k+l} = \binom{n+k}{k}\binom{n+l}{l}.$$

which is a generalisation of an identity found by the ancient Chinese mathematician Chu Shih-Chieh. The first proof, due essentially to Hua Loo-Keng, uses algebraic means. The second applies to the form

$$\sum_{i=0}^{k} P(k-i, l-i, n-i, i, i) = P(k, l)P(l, n)P(n, k),$$

$$P(k, l) = \frac{(k+l)!}{k!l!},$$

of the identity, expressed by numbers of permutations with equal elements, and proves the identity by finding appropriate combinatorical meanings for both of the sides. *Author's summary.*

Huszár, Géza. On a problem in the history of Chinese mathematics. Mat. Lapok **6** (1955), 36–38. (Hungarian. Russian and English summaries)

The author treats the peculiar formula

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k} = \binom{n+k}{k}^2$$

of the old mathematics of the Chinese. He proves the

same formula by applying some elementary properties of the binomial coefficients. *Author's summary.*

Carlitz, L. On a problem of the history of Chinese mathematics. Mat. Lapok **6** (1955), 219–220. (Hungarian. English and Russian summaries)

"···I was interested in the formula of Le-Jen Shoo:

$$(*)\quad \sum_{j=0}^{k}\binom{k}{j}^2\binom{n+2k-j}{2k}=\binom{n+k}{k}^2$$

recently proved by you in Mat. Lapok **5** (1954), 1–6 [MR **16**, 13]. It may be of interest to note that (*) can be proved rapidly in the following way. Using the standard notation of hypergeometric series, the left member is evidently equal to

$$(1)\quad \binom{n+2k}{2k}\sum_{j=0}^{k}\frac{(-k)_r(-k)_r(-n)_r}{r!r!(-n-2k)_r}=\binom{n+2k}{2k}\,{}_3F_2\left[\begin{matrix}-k,\ -k,\ n\\ 1,\ -n-2k\end{matrix}\right].$$

Summing ${}_3F_2$ by Saalschütz's theorem [see, for example, Bailey's Generalised hypergeometric series, Cambridge, 1935, p. 9] we get

$$(2)\quad \binom{n+2k}{2k}\frac{(1+k)_n(1+k)_n}{(1)_n(1+2k)_n}=\frac{(n+2k)!}{n!(2k)!}\,\frac{(n+k)!(n+k)!(2k)!}{n!k!k!(n+2k)!}=\binom{n+k}{k}^2.$$

This evidently completes the proof of (*)···". Finally, the editor recalls the definition of the generalised hypergeometric series and the proof of Saalschütz's theorem. *English summary.*

Kanold, Hans-Joachim. Vollkommene und befreundete Zahlen. Nachr. Giessener Hochschulges. **24** (1955), 122–130.

Expository paper. *I. Niven* (Berkeley, Calif.).

Hornfeck, Bernhard. Zur Dichte der Menge der vollkommenen Zahlen. Arch. Math. **6** (1955), 442–443.

The number n is perfect if $\sum_{d|n}d=2n$. The author proves that there are less than $\sqrt{x}$ perfect numbers $\leqq x$. This improves a result by Kanold [Math. Z. **61** (1954), 180–185; MR **16**, 337]. *P. Scherk.*

Erdös, Paul. Some remarks on number theory. Riveon Lematematika **9** (1955), 45–48. (Hebrew. English summary)

The following theorems are established. (1) There exists a set of real numbers $\{a_t\}$ of cardinality c so that all the sequences $\{[a_t{}^n]\}_{n=0}^{\infty}$ are "far apart" in the sense of Hartman, that is the inequality $|[a_t{}^n]-[a_{t'}{}^m]|<A$ has only a finite number of solutions n, m for every $t\neq t'$ and every A. (2) Let $A(n)$ be the number of integers which are the product of two integers no greater than n. Then $A(n)=o(n^2/\log^{\alpha}n)$ for some $\alpha>0$.

The following questions are raised. (1) Is there a field (or even ring) of real numbers of cardinality c which contains no Liouville numbers? (2) Let the integers n of the interval $(1, 4n)$ be divided into the disjoint classes $a_1, a_2, \cdots, a_{2n}$ and $b_1, b_2, \cdots, b_{2n}$. Does there exist an integer t so that the number of solutions of $a_i+t=b_j$ is no less than n? Scherk has established that this number is $>(2-\sqrt{2})n$ for some t. (3) Find the least upper bound for the exponent α in theorem (2) above. (4) Given integers $1\leqq a_1<a_2<\cdots<a_s\leqq n$, $1\leqq b_1<b_2<\cdots<b_v\leqq n$ so that the products a_ib_j are distinct. Is it true that $xy<cn^2/\log n$?

There are several misprints: On p. 45, l. 22 read a_{t_i} for b_{t_i}; p. 45, l. 30 read $u_n=3^n$; p. 46, l. 6 read $b_{n-1}\leqq q<b_n$; eqn. (6) and p. 47, l. 1 read 9 for q; p. 47, l. 33 read $(4/3-\varepsilon)\log\log(n^2)$; p. 48, l. 1 read $o(n^2/\log^{\alpha}n)$.

E. G. Straus (Princeton, N.J.).

Nöbauer, Wilfried. Über die Formengruppe. Monatsh. Math. **59** (1955), 305–317.

The terminology is that of a previous paper of the author [Monatsh. Math. **59** (1955), 118–145; MR **16**, 906] and is explained in the review of that paper. A polynomial vector of Γ_k is called a form-vector of degree t if every component is a form of degree t. The form-vectors constitute a subgroup $\mathfrak{F}_n{}^k$ of the previously defined group $\mathfrak{G}_n{}^k$; $\mathfrak{F}_n{}^k$ is called a form-group and is investigated in the present paper. It is pointed out that the case of quadratfrei n is of primary interest since only in this case do form-vectors of degree >1 occur. The principal results of the present paper are the following. I. Definition and existence of $\mathfrak{F}_n{}^k$. II. If n is not quadratfrei, then $\mathfrak{F}_n{}^k$ is identical with the group $\mathfrak{R}_n{}^k$ consisting of elements of $\mathfrak{G}_n{}^k$ with representative vectors of the type $\mathfrak{g}(\mathfrak{x})=A\mathfrak{x}$, where A is a square matrix of order k. III. The group $\mathfrak{F}_n{}^k$ can be mapped homorphically on the group (mod $\bar{n}$) of residues prime to $\bar{n}$, where $\bar{n}=\text{l.c.m.}[p-1]$, $p|n$. IIIa. The elements of $\mathfrak{F}_n{}^k$ whose form-vectors $\mathfrak{f}(\mathfrak{x})$ satisfy $[\mathfrak{f}]\equiv 1 \pmod{\bar{n}}$, constitute a normal subgroup $\mathfrak{E}_n{}^k$. The factor group $\mathfrak{F}_n{}^k/\mathfrak{E}_n{}^k$ is isomorphic to the group of residues (mod $\bar{n}$). IV. A group $\mathfrak{B}_n{}^k$ is defined; the precise definition is too involved to be given here. It is proved that if $n=ab$, where $(a, b)=1$, then $\mathfrak{B}_n{}^k\cong\mathfrak{B}_a{}^k\times\mathfrak{B}_b{}^k$ and if n is quadratfrei, then also $\mathfrak{E}_n{}^k\cong\mathfrak{E}_a{}^k\times\mathfrak{E}_b{}^k$. V. When n is quadratfrei, the groups $\mathfrak{B}_n{}^k$ and $\mathfrak{E}_n{}^k$ are identical. VI. If p is a prime, the group $\mathfrak{E}_p{}^k$ is isomorphic to the symmetry $\Sigma_r(Z_{p-1})$, where Z_{p-1} is the cyclic group of order $p-1$ and $r=(p^k-1)/(p-1)$. VII. If $\mathfrak{R}_n{}^k$ has the same meaning as in II above, then $\mathfrak{E}_n{}^k=\mathfrak{R}_n{}^k$ for $k=1$ and quadratfrei n, also for $k=2$, $n=2$, but in no other cases. *L. Carlitz.*

Carlitz, L. Some partition formulas related to sums of squares. Nieuw Arch. Wisk. (3) **3** (1955), 129–133.

Let $\vartheta_4(q)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}$. Extending the method of his earlier paper [Quart. J. Math. Oxford Ser. (2) **4** (1953), 168–172; MR **15**, 201] the author shows that identities such as

$$(*)\quad 1+16\sum_{n=1}^{\infty}\frac{n^3q^n}{1-(-1)^nq^n}=\vartheta_4{}^8(q)$$

can be obtained from the familiar formula $\wp'(u)=-\sigma(2u)/\sigma^4(u)$, when the $\wp$-function is replaced by its Fourier series. Evidently (*) yields the well-known formula for the number of representations of an integer as the sum of 8 squares. *A. L. Whiteman* (Los Angeles, Calif.).

Carlitz, L. The number of solutions of a special quadratic congruence. Portugal. Math. **14** (1955), 9–14.

Let p be a prime >2 and r, t be fixed integers $\geqq 1$. Let $A=(a_{ij})$ denote a symmetric matrix of order m, where the a_{ij} are integers such that $|A|\not\equiv 0 \pmod{p}$. For a fixed integer b let $N_t(b)=N_t(A, b, t)$ denote the number of incongruent solutions X of the congruence

$$(*)\quad \begin{vmatrix}A & X\\ X' & 0\end{vmatrix}\equiv b \pmod{p^r},$$

where $X=(x_{ij})$ is an $m\times t$ matrix and X' is the transpose

of X. Transforming (*) into an equivalent form and then applying some results of Siegel [Ann. of Math. (2) **36** (1935), 527–606] on the number of solutions of certain matric congruences, the author expresses $N_q(b)$ in terms of simple product formulas. *A. L. Whiteman.*

Palamà, G. Congruenze multigrade. Period. Mat. (4) **33** (1955), 230–234.

Kesava Menon, P. On the equation $X^3+Y^3=U^3+V^3$. Math. Student **23** (1955), 101–103.

All solutions in integers of $X^3+Y^3=U^3+V^3$ are obtained explicitly in terms of four unrestricted integer parameters. We omit the details. The method involves the quadratic field generated by the cube roots of unity. [The following minor error occurs in the final solution: equations (8) lack the factor 3 in the terms qr and ps.] *I. Niven* (Berkeley, Calif.).

Schinzel, A. On functions $\varphi(n)$ and $\sigma(n)$. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 415–419.

Let $a_1, a_2, \cdots, a_h$ be any finite sequence of non-negative numbers or infinity. Then there exists an infinite sequence $n_1<n_2<n_3<\cdots$ of natural numbers such that

$$\lim_{k\to\infty}\frac{\phi(n_k+i)}{\phi(n_k+i-1)}=a_i \text{ for } i=1, 2, \cdots, h,$$

$\phi(n)$ being the Euler function. The same result holds for the sum of divisors function $\sigma(n)$. These results generalize previous work by the author [same Bull. **2** (1954), 467–469; MR **16**, 675] and by the author and W. Sierpiński [ibid. **2** (1954), 463–466; MR **16**, 675]. *I. Niven.*

Halberstam, H. On the distribution of additive number-theoretic functions. II. J. London Math. Soc. **31** (1956), 1–14.

Let f be a function defined initially on the primes, and for m and n positive integers put

$$f(m)=\sum_{p|m} f(p),$$

$$A_n=\sum_{p<n}\frac{f(p)}{p}, \quad B_n=\sum_{p<n}\frac{f^2(p)}{p}.$$

Erdös [Ann. of Math. (2) **47** (1946), 1–20; MR **7**, 416] pointed out that a method developed by him and Kac [Amer. J. Math. **62** (1940), 738–742; MR **2**, 42] could be used to prove the following theorem: Let $k_n(\omega_1, \omega_2)$ be the number of positive integers $m\leq n$ for which simultaneously

$$f(m)<A_n+\omega_1 B_n^{\frac{1}{2}}, \quad f(m+1)<A_n+\omega_2 B_n^{\frac{1}{2}},$$

and let $s_n(\omega)$ be the number of such integers for which

$$f(m)<f(m+1)+\omega(2B_n)^{\frac{1}{2}}.$$

Then if $|f(p)|\leq 1$ for all p and $B_n\to\infty$ with n,

$$\lim_{n\to\infty} n^{-1}k_n(\omega_1, \omega_2)=\phi(\omega_1)\phi(\omega_2),$$

$$\lim_{n\to\infty} n^{-1}s_n(\omega)=\phi(\omega),$$

where

$$\phi(\xi)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\xi} e^{-u^2/2}du.$$

The proof, using Brun's method and the central-limit theorem, was given by the reviewer [Trans. Amer. Math. Soc. **66** (1949), 440–463; MR **11**, 83]. In the present paper the author uses his method of moments [J. London Math. Soc. **30** (1955), 43–53; MR **16**, 569] to strengthen this theorem; the requirement $|f(p)|\leq 1$ is replaced by $f(p)=o(B_p^{\frac{1}{2}})$.

The same method is used to show that the frequency of solutions $m\leq n$ of the inequality $f(g(m))<A(n)+\omega(B(n))^{\frac{1}{2}}$ is also asymptotically normal; here g is an irreducible polynomial with integral coefficients,

$$A(n)=\sum_{p<n}\frac{f(p)r(p)}{p}, \quad B(n)=\sum_{p<n}\frac{f^2(p)r(p)}{p},$$

and $r(a)$ is the number of solutions of $g(m)\equiv 0 \pmod a$, $0\leq m<a$. *W. J. LeVeque* (Ann Arbor, Mich.).

Halberstam, H. On the distribution of additive number-theoretic functions. III. J. London Math. Soc. **31** (1956), 14–27.

Let f be an irreducible polynomial with integral coefficients, and let $\omega(m)$ be the number of distinct primes dividing m. Improving a result of Prachar [same J. **28** (1953), 236–239; MR **14**, 726], it is shown that

$$\sum_{p\leq m}\omega(f(p))\sim\frac{n\log\log n}{\log n}$$

and the same method shows that for all but $o(n/\log n)$ primes $p\leq n$, $\omega(f(p))=\log\log n$. By a result on moments similar to those mentioned in the preceding review, a general theorem is proved which has the following as a special case: If $\varkappa_n(\vartheta)$ is the number of primes $p\leq n$ for which

$$\omega(f(p))<\log\log n+\vartheta(\log\log n)^{\frac{1}{2}},$$

then

$$\lim_{n\to\infty}\frac{\varkappa_n(\vartheta)}{\pi(n)}=\phi(\vartheta).$$

The proof of this last theorem uses a deep result on primes in an arithmetic progression. *W. J. LeVeque.*

Erdös, P. On consecutive integers. Nieuw Arch. Wisk. (3) **3** (1955), 124–128.

Define $f(k)$ as the least integer such that the product of $f(k)$ consecutive integers, each greater than k, is always divisible by a prime greater than k. A theorem of Sylvester and Schur [for a proof as well as for references to the papers of Sylvester and of Schur, see Erdös, J. London Math. Soc. **9** (1934), 282–288] states that $f(k)\leq k$. The exact determination of the order of $f(k)$ is very difficult, but the author now takes a first step towards it by proving that there exists a constant $c>1$ such that $f(k)\leq ck(\log k)$; he conjectures that the true order of magnitude of $f(k)$ is more likely to be comparable with the largest difference $p_{r+1}-p_r$, $k<p_r<p_{r+1}<2k$. The author goes on to prove that if $g(k)$ is the smallest integer such that at least $g(k)$ among k consecutive integers, each greater than k, have prime factors greater than k, $g(k)=\{1+o(1)\}k/(\log k)$. The paper concludes with a third, related, result.

The principal arguments are elementary and stem from the prime decomposition of the binomial coefficient $\binom{n+t}{t}$, $n>k$, $t\leq k$, but it is found necessary to use also the Prime Number Theorem and a deep result about gaps between consecutive primes.

H. Halberstam (Providence, R.I.).

Erdös, P. Über die Anzahl der Lösungen von $[p-1, q-1]\leq x$. (Aus einem Brief von P. Erdös an K. Prachar.) Monatsh. Math. **59** (1955), 318–319.

It is proved that the number of solutions of $[p-1,$

$q-1]\leqq x$, where p, q are primes and the left member denotes the least common multiple, is less than $cx(\log\log x)^3$. [Cf. K. Prachar, Monatsh. Math. **59** (1955), 91–97; MR **16**, 904.] The proof makes use of Brun's sieve method. In a note added in proof it is pointed out that the number of solutions is indeed less than cx.

L. Carlitz (Durham, N.C.).

★**Dehn, Edgar. Prime numbers. A study of their distribution.** Rock Way House, Yonkers, N. Y., 1952. viii+48 pp.

The author sets out to discuss several well-known problems in prime number theory, such as Goldbach's conjecture and the estimation of gaps between consecutive primes, but his intentional neglect of all error terms and cavalier use of the equality sign render most of the arguments invalid, and, indeed, prevent him from preceiving the real difficulties in each problem. In particular, the key formulae on page 10, line 17, page 12, line 18 and page 13, line 5, are false. Some numerical evidence of a restricted nature is put forward in support of the author's conclusions.

The book ends with two letters on freedom of scientific thought.

H. Halberstam (Providence, R.I.).

See also: Sonneborn, p. 459.

Analytic Number Theory

Min, Szu-Hoa. On a way of generalization of the Riemann ζ function. I. The analytic continuation of $Z_{n,k}(s)$ to the whole plane. Acta Math. Sinica **5** (1955), 285–294. (Chinese. English summary)

Let n be even, $\nu=1/n$, $s=\sigma+it$, and

$$Z_{n,k}(s)\sum_{a_1,\cdots,a_k=-\infty}^{+\infty}{}'(x_1^n+\cdots+x_k^n)^{-s}$$

(the dash means that there is no term $x_1=\cdots=x_k=0$). The author proves that $Z_{n,k}(s)$ is meromorphic, with one simple pole of residue $2^k\Gamma^k(1+\nu)/\Gamma(k\nu)$ at $s=k\nu$, by showing that

$$Z_{n,k}(s)=\frac{1}{\Gamma(s)}\Big\{\frac{2^k\Gamma^k(1+\nu)}{s-k\nu}-\frac{1}{s}$$
$$+\int_1^\infty\omega^{s-1}\Big(\Big(\sum_{a=-\infty}^{+\infty}e^{-a^n\omega}\Big)^k-1\Big)d\omega+\int_0^1 2^k\omega^{s-k\nu-1}\Big[\Big(\Gamma(1+\nu)$$
$$+2\omega^{2k\nu}\sum_{\nu=1}^{\infty}\frac{(-1)^k}{(2\pi y)^{2k}}\int_1^\infty\frac{d^{2k}e^{-x^n}}{dx^{2k}}\cos\frac{2\pi yx}{\omega^\nu}\,dx\Big)^k$$
$$-\Gamma^k(1+\nu)\Big]d\omega\Big\}.$$

Here k can be any positive integer; a slightly different development holds when $k=0$. A much more general theorem was already obtained by Hj. Mellin [Acta Soc. Sci. Fenn. **29** (1902), no. 4].

K. Mahler.

Tong, Kwang-Chang. On division problems. I. Acta Math. Sinica **5** (1955), 313–324. (Chinese. English summary)

Let $d_k(n)$ be the number of positive integral solutions of $n=n_1\cdots n_k$; let $D_k(x)=\sum_{n\leqq x}d_k(n)$; let

$$R_k(x)=x(a_{k0}+a_{k1}\log x+\cdots+a_{k,k-1}\log^{k-1}x)$$

be the residue of $\zeta(s)^kx^s/s$ at $s=1$; and let $\Delta_k(x)=D_k(x)-R_k(x)$. The author proves the following results. (1) Let $N\geqq0$; let the integer h be large compared with $k/2$; and let $\min(x, x+hy)>0$. Then

$$\int_0^y\cdots\int_0^y\Delta_k(x+y_1+\cdots+y_h)dy_1\cdots dy_h=\frac{y^h}{(-2)^k}$$
$$+\sum_{n\leqq N}d_k(n)\int_0^y\cdots\int_0^y I_{k,0}(n,x+y_1+\cdots+y_h)dy_1\cdots dy_h$$
$$+\sum_{l=0}^{h}(-1)^{h-l}\binom{h}{l}\sum_{n>N}^{h}d_k(n)I_{k,h}(n,x+ly).$$

Here

$$I_{k,j}(n,x)=\frac{2^k}{2\pi i}\int_{C_j}\frac{x^{j+1-s}}{(2\pi)^{ks}n^s}\frac{(\Gamma(s)\cos\frac{1}{2}\pi s)^k}{(1-s)(2-s)\cdots(j+1-s)}ds$$
$$(x>0;\ n=1,2,3,\cdots),$$

where C_j, $j=0,1,2,\cdots$, is the polygon with vertices at

$$-i\infty,\ -i,\ j+\frac{3}{2}-i,\ j+\frac{3}{2}+i,\ i,\ i\infty.$$

(2) If nx is large, then

$$I_{k,j}(n,x)\sim\sum_{m=0}^{\infty}c_{kjm}x^{j+1}(nx)^{-\frac{(k+1)}{2k}-\frac{(j+m)}{k}}\cos\{2k\pi(nx)^{1/k}+\theta_{kjm}\}$$

where $c_{kj0}=(2^j\pi^{1+j}\sqrt{k})^{-1}$, $\theta_{kj0}=\frac{1}{2}\pi(\frac{1}{2}(k-3)-j)$, and c_{kjm} and θ_{kjm} are real numbers. From these two lemmas, and with the help of a further asymptotic result, the following two main theorems are deduced: (3) There exist two positive constants c_k and C_k such that, if $x\geqq1$ and $-C_kx^{(k-1)/2k}\leqq t\leqq C_kx^{(k-1)/2k}$, there is at least one y satisfying $x\leqq y\leqq x+c_kx^{1-1/k}$ and $\Delta_k(y)=t$. (4) There exists a positive constant $\tilde{C}_k$ such that

$$\Big\{\frac{1}{x}\int_0^x|\Delta_k(y)|^kdy\Big\}^{1/k}\geqq\tilde{C}_kx^{\frac{k-1}{2k}}$$

for every $\lambda\geqq1$ and every $x\geqq1$. [See G. Voronoi, Ann. Sci. Ecole norm. Sup. (3) **21** (1904), 207–267, 459–533; G. H. Hardy, Proc. London Math. Soc. (2) **15** (1916), 1–25; E. C. Titchmarsh, The theory of the Riemann zeta-function, Oxford, 1951, ch. 12, in particular p. 272; MR **13**, 741.]

K. Mahler (Manchester).

Knapowski, S. On the greatest prime factors of certain products. Ann. Polon. Math. **2** (1955), 56–63.

Generalizing theorems of Chebyshev, Ivanov and Størmer [E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Bd. 1, Teubner, Leipzig-Berlin, 1909, pp. 559–564] the writer proves the following results. I. Let $\{a_n\}$ be a sequence of integers $0<a_1<a_2<\cdots$ and A a positive integer. Let P_n denote the greatest prime factor of $\prod_1^n(A+a_n^2)$. If

$$\liminf_{n=\infty}\frac{\log(a_1\cdots a_n)}{a_n\log a_n}>\tfrac{1}{2},$$

then $\lim_{n=\infty}P_n/a_n=\infty$; and if

$$\liminf_{n=\infty}\frac{\log(a_1\cdots a_n)}{a_n\log n}>\tfrac{1}{2},$$

then $\lim_{n=\infty}P_n/x=\infty$. II. Let $\{a_n\}$ be a sequence of integers $0<a_1<a_2<\cdots$ and let either

$$\liminf_{n=\infty}\frac{\log(a_1\cdots a_n)}{a_n\log a_n}>\tfrac{1}{2}$$

or

$$\liminf_{n=\infty}\frac{\log(a_1\cdots a_n)}{a_n\log n}>\tfrac{1}{2}\text{ and }a_n=O(n).$$

Please insert this memorandum in Mathematical Reviews, May issue 1956, page 462, and in Prime Numbers by Edgar Dehn, 1952.

Mathematical Reviews, executive editor John V. Wehausen, published this "review" by H. Halberstam (Providence, R. I.) of Prime Numbers by Edgar Dehn, 1952, in its May issue 1956 which appeared in August:

The author sets out to discuss several well-known problems in prime number theory, such as Goldbach's conjecture and the estimation of gaps between consecutive primes, but his intentional neglect of all error terms and cavalier use of the equality sign render most of the arguments invalid, and, indeed, prevent him from perceiving the real difficulties in each problem. In particular, the key formulae on page 10, line 17, page 12, line 18, and page 13, line 5, are false. Some numerical evidence of a restricted nature is put forward in support of the author's conclusions.

The book ends with two letters on freedom of scientific thought.

Mathematical Reviews is now requested to print this rejoinder by Edgar Dehn:

Without any reasons named that could fit each case, three "key formulae" of my book are adjudged "false" in the review.

The first of these is no key formula, being used in the book nowhere at all. Together with the preceding formula, which was perhaps meant although called not new in my book, it may be considered a special case of a classic theorem on indicators which goes back, I think, to Legendre. If all that is false, it is just too bad.

The other two formulae involve errors, to be sure, and so does

my real key formula which stands between them. But these errors are limited by the coefficient k that links my formula to the prime number theorem, and does so by two different means.

My work did not require a closer treatment of errors, but I explained when and why they should be small. They render no "arguments invalid" because they either are not involved, or there is room for them if they are, or they cancel out. And no equality signs were used to conceal them, as the review implies.

What calculations of errors I possess discloses nothing that was not obvious before. Neither does it suggest that my errors ever become as great as k would permit if all implications of accepted theory were correct.

Numerical evidence is of necessity "restricted" in my book, yet could it make anybody pause who does not wilfully choose to be blind.

ROCK WAY HOUSE, BOX 222 SOUTH, YONKERS 5, NEW YORK

Then among the numbers $i(i-a_1)\cdots(i-a_n)$ at most a finite number are real or pure imaginary. In addition two further theorems are proved. *L. Carlitz.*

Algebraic Number Theory

Dufresnoy, J., et Pisot, C. Sur les éléments d'accumulation d'un ensemble fermé d'entiers algébriques. Bull. Sci. Math. (2) **79** (1955), 54–64.

Let S be the set of algebraic integers θ all of whose conjugates except θ itself lie interior to the unit circle, and let S' be the derived set of S. It is shown that the only elements of S' numerically smaller than 1.8 are $(1+5^{\frac{1}{2}})/2=1.61\cdots$ and $1.75\cdots$, these being zeros of the polynomials $1+z-z^2$ and $1-z+2z^2-z^3$ respectively. The proof uses a characterization of S' obtained earlier by the authors [Ann. Sci. Ecole Norm. Sup. (3) **70** (1953), 105–133; MR **15**, 605] and a method recently developed for them for investigating S [ibid. **72** (1955), 69–92; MR **17**, 349].

W. J. LeVeque (Ann Arbor, Mich.).

Rieger, G. J. Zum Waringschen Problem fur algebraische Zahlen and Polynome. J. Reine Angew. Math. **195** (1956), 108–120 (1955).

Kamke [Math. Ann. **87** (1922), 238–245] proved the following two theorems.

I. For each positive integer n there exists an integer $N_1=N_1(n)$ with the following property: Let K be an algebraic number field and

$$\phi(x)=\sum_{r=0}^{n}\alpha_r x^{n-r}\quad(\alpha_r\in K,\ \alpha_0>0)$$

a polynomial with at least one real root in each real conjugate field $K^{(i)}$. Let $u^{(i)}$ denote the largest real root in $K^{(i)}$. Then for every $\xi\in K$, $\xi>0$, the equation

$$\xi=\sum_{r=1}^{N_1}\phi(\xi_r)\quad(\xi_i\in K)$$

is solvable with $\xi_r^{(i)}>u_i$ in each real $K^{(i)}$. (By $\alpha>\beta$ is meant $\alpha^{(i)}>\beta^{(i)}$ in each real $K^{(i)}$).

II. For each n there exists an $N_2=N_2(n)$ with the following property: If $\gamma(x)$ is a rational function with rational coefficients of degree μn (μ even), then there exist N_2 rational functions with rational coefficients $\xi_r(x)$ with non-vanishing denominators such that

$$\gamma(x)=\sum_{r=1}^{N_2}\xi_r^{n}(x)\quad(\xi_r(x)\geqq 0).$$

The object of the present paper is to find upper bounds for N_1 and N_2. Using the elementary method developed in his dissertation [see Arch. Math. **4** (1953), 275–281; MR **15**, 289], the author proves that

$$N_1=N_1(n)<(2n+3)^{3(2n+3)^n},$$
$$N_2=N_2(n)<(2n+3)^{3(2n+3)^{2n}}.$$

It is remarked that the result of his dissertation concerning $g(n)$ in Waring's problem can be improved to

$$g(n)<(2n+1)^{200(n+1)^n}.$$

L. Carlitz (Durham, N.C.).

Carlitz, L., and Corson, H. H. Some special equations in a finite field. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 752–754.

The following is a typical theorem of the authors: Let $N(a_1,\cdots,a_r)$ be the number of solutions of

$$a_1x_1^{m_1}+\cdots+a_rx_r^{m_r}=0,\quad m_i\mid(q-1),$$

in the finite field GF(q) with $q=p^n$ elements. The average of $N(a_1,\cdots,a_r)$ is evidently equal to $A=q^{r-1}$. Let $\lambda(i_1,\cdots,i_k)=m_{i_1}\cdots m_{i_k}/[m_{i_1},\cdots,m_{i_k}]$ where $[\ ,\cdots,\ ]$ denotes the least common multiple of the numbers indicated. Then the variance

$$\sum_{a_i\neq 0}(N(a_1,\cdots,a_r)-A)^2$$
$$=q^{r-2}(q-1)^{r+2}\sum_{k=0}^{r}\sum_{i}(-1)^{r-k}\lambda(i_1,\cdots,i_k),$$

where the inner sum is over the r indices taken k at a time, and $\lambda(i_1,\cdots,i_k)=1$ for $k=0$. *L. K. Hua.*

Tornheim, Leonard. Minimal basis and inessential discriminant divisors for a cubic field. Pacific J. Math. **5** (1955), 623–631.

The author finds values of integers A, B, D in the integral basis 1, θ, $(\theta^2+A\theta+B)/D$ for any cubic field over the rationals using generators of the type $\theta^3+a\theta^2+b=0$ with integral a, b. [Cf. Albert, Ann. of Math. (2) **31** (1930), 550–566.] Thus the field discriminant D^* is $D(\theta)/D^2$, where $D(\theta)=\prod_{i>j}(\theta_i-\theta_j)^2=b(4a^3+27b)$. With this basis he shows that if α is an arbitrary integer of the field then g.c.d. $D(\alpha)=D^*$ or $4D^*$, whence $4^{\frac{1}{3}}$ is the only non-trivial inessential discriminant divisor F. This more than verifies (without ideal-theory) a theorem of Żyliński [Math. Ann. **73** (1913), 273–274] which asserts that F has only those primes less than the degree as possible factors.

H. Cohn (Detroit, Mich.).

Vandiver, H. S. On the properties of certain trinomial equations in a finite field. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 651–653.

Let g be a multiplicative generator of the cyclic group formed by the non-zero elements of the finite field $F(p^n)$ of order p^n. The author considers the number of solutions (s, t) of the equation $g^{t+cs}+g^{s+mt}=1$, where $p^n-1=cm$, $(c, m)=1$ $(0\leq s<m,\ 0\leq t<c)$. He derives certain quadratic relations between these numbers and in certain cases is able to deduce that the equation has no solution.

H. W. Brinkmann (Swarthmore, Pa.).

Vandiver, H. S. Relation of the theory of certain trinomial equations in a finite field to Fermat's last theorem. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 770–775.

By the use of a theorem of reciprocity the author relates the study of the possibility of the Fermat equation $x^l+y^l+z^l=0$ (l an odd prime) to the study of equations of the form $g^{t+cs}+g^{s+ct}=1$ in a finite field $F(p^n)$; here $p^n=1+cl$, $c\equiv 0 \pmod l$, g is a multiplicative generator of the nonzero elements of $F(p^n)$ and $0\leq s<l$, $0\leq t<c$. The study of Fermat's last theorem can then be continued by making use of the author's many relations between the number of solutions, s, t of the equation just given.

H. W. Brinkmann (Swarthmore, Pa.).

Vandiver, H. S. On cyclotomic relations and trinomial equations in a finite field. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 775–780.

Let $F(p^n)$ be a finite field of order p^n, let $p^n-1=cq$, where q is an odd prime and $c\not\equiv 0 \pmod q$; also let g be a multiplicative generator of the nonzero elements of $F(p^n)$. The author considers the equation $g^{t+cs}+g^{s+ct}=1$, where $0\leq s<q$, $0\leq t<c$. He sets (i, j) equal to the number of

solutions s, t of his equation and derives relations between these numbers, mainly by the use of cyclotomic relationships. His main result consists in evaluating the sum $\sum_{i=0}^{q-1}(i, j)(i-m, j+d)$ under the assumption that there exists, for each fixed i, a j such that $(i, j)=q$. The possible values of the sum in question are very simple and are of the form $c-x$ where $x=0, 1, q, q+1$. *H. W. Brinkmann.*

Vandiver, H. S. On the divisors of the second factor of the class number of a cyclotomic field. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 780–783.

Let $\zeta=e^{2\pi i/l}$ where l is an odd prime and let $h_2(l)$ be the second factor of the class number of the field $K(\zeta)$. The author devises an algebraic procedure to find out if $h_2(l)$ is divisible by l for a given l. *H. W. Brinkmann.*

Aigner, Alexander. Die kubische Fermatgleichung in quadratischen Körpern. J. Reine Angew. Math. **195** (1956), 3–17 (1955).

The author continues his studies [Monatsh. Math. **56** (1952), 240–252, 335–338; MR **14**, 452, 621] whether quadratic extensions $K(m^{\frac{1}{3}})$ of the rational field (m= square-free rational integer) contain a solution of (*) $x^3+y^3=z^3$, and, in particular, he shows how to construct numbers k such that (*) is insoluble for all $m=\pm Rk$ where $R=1$ or contains only prime factors of which k is a cubic non-residue. As he has shown (loc. cit.), one can assume $3\nmid m$ and a solution exists if and only if there is a rational integer solution of $X(X^3+4Y^3)=mZ^3$, $(X, Y)=1$ [this is substantially a special case of Nagell's theorem that $4x^3-g_2x-g_3=y^2$ (g_2, g_3 rational) is soluble in $K(m^{\frac{1}{3}})$ if and only if $4x^3-g_2x-g_3=my^3$ is soluble in rationals, Nova Acta Soc. Sci. Upsal. (4) **13** (1942), no. 3; MR **8**, 315]. Hence $X^3+4Y^3=m_1Z_1^3$, $X=m_2Z_2^3$ where $m_1m_2=m$, $Z_1Z_2=Z$ and m_1 is odd and positive; and so

$$m_2Z_2^3+2^3Y=\alpha\xi^3$$

where $\alpha, \xi \in K(2^{\frac{1}{3}})$ and norm $\alpha=m_1$. Here $\alpha=1$ leads to an infinite descent and the remaining cases may often be excluded by taking congruences to the prime divisors of m in $K(2^{\frac{1}{3}})$. The author develops a systematic process for this. He works largely by expressing the numbers of $K(2^{\frac{1}{3}})$ in terms of the basis $1, 2^{\frac{1}{3}}, 2^{\frac{2}{3}}$ and then applies the law of quadratic reciprocity in the rational field, but the reviewer suspects that some of the ingenuity would not be necessary if he had used the reciprocity law for $K(2^{\frac{1}{3}})$. The author concludes with the (rash) conjecture that (*) is soluble in all m not excluded by the criteria of this and previous papers and, in particular, when $|m|=2$ (3).

J. W. S. Cassels (Cambridge, England).

Skolem, Th. Einige Bemerkungen über die Auffindung der rationalen Punkte auf gewissen algebraischen Gebilden. Math. Z. **63** (1955), 295–312.

There are, first of all, a few remarks on rational points upon a rational unicursal curve, and others concerning the points of such a curve belonging to an algebraic field and the existence of parametric solutions, accompanied by a number of interesting examples, where the rational points are given by transcendental functions of a parameter varying in a module. The rational points on a cubic rational surface are then studied, considering both the singular non-ruled surfaces and the non-singular ones. Particularly noteworthy is the proof for the singular cubic surfaces of Mordell's conjecture [Publ. Math. Debrecen **1** (1949), 1–6; MR **11**, 82], according to which the equation $f(x, y, z, u)=0$, where f is a cubic form, has rational solutions if the congruence $f(x, y, z, u)\equiv 0$ is solvable for every module.

Also a very simple proof is given of the theorem [B. Segre, J. London Math. Soc. **18** (1943), 226–233; MR **6**, 37] which affirms that every rational cubic surface containing a rational simple point has an infinity of these points. The result is obtained by a geometric construction, which is then applied for solving Ryley's equation $x^3+y^3+z^3=r$; by another simple method, solutions of

$$xy(x+y)+z^3=r,\quad x^3+y^3+z^3=r,\quad x^3+y^3+z^3=r$$

are also deduced. Finally, the cubic equation

$$x^3+2y^3+3z^3+6u^3=0$$

is solved completely in the rational field, by applying the transformation

$$x:y:z:u=-t_0t_1+2t_2^2:-t_1^2+t_0t_2:-t_1t_2:t_2t_3,$$

which reduces it to the cubic equation

$$t_0^3+2t_1^3+4t_2^3-6t_0t_1t_2+3t_3^3=0;$$

the success of the method is due to the fact that, while the first cubic surface is non-singular, the second one possesses three (irrational cubic conjugate) double points; and the author asks whether the method can be extended. [The reviewer will give an answer to this question in a forthcoming paper.] *B. Segre* (Rome).

Whaples, G. Generalized local class field theory. II. Existence theorem. Duke Math. J. **21** (1954), 247–255.

[For part I see same J. **19** (1952), 505–517; MR **14**, 140.] Let k be a field complete with respect to a discrete valuation whose residue field $[k]$ is perfect and has exactly one extension of degree n for each natural number n. Let π be a prime element in the ring of integers $\mathfrak{o}$ of k. A multiplicative subgroup a of k is called analytic if it has a conductor π^{e+1} and if for every $i>0$ there exists a polynomial $u_i(x) \in \mathfrak{o}[x]$ such that $1+u_i(\xi)\pi^i \in a$ for all $\xi \in \mathfrak{o}$ and such that its residue polynomial $[u_i(x)]$ has positive degree. Using results from the paper reviewed fourth below, Whaples shows first that the $u_i(x)$ can be chosen so that the $[u_i(x)]$ are additive polynomials in $[k][x]$. He then shows that the intersection of two analytic groups is analytic, that the inverse image of an analytic group under the norm from a pure ramified extension is analytic in the top field, and that the group of norms from any separable extension is analytic. By explicitly constructing cyclic fields with sufficiently general norm groups, he then obtains enough information to prove his existence theorem: The norm groups of abelian extensions are the analytic subgroups of finite index. The gap in the beginning of the inductive argument in the proof of the main theorem is acknowledged in the last sentence of the paper. *J. Tate* (Cambridge, Mass.).

Whaples, G. Generalized local class field theory. III. Second form of existence theorem. Structure of analytic groups. Duke Math. J. **21** (1954), 575–581.

With the same notations as in the preceding review, let w be the ring of Witt vectors with residue field $[k]$. To each $\alpha=\sum \alpha_s p^s \in w$ there is attached a formal power series $E(\alpha, x)=\prod(F(\alpha_s x))^{p^s}$ with coefficients in $\mathfrak{o}$, where $F(y)=\prod(1-y^a)^{\mu(a)/a}$, product over all a not divisible by p. The map $\alpha \to E(\alpha, x)$ has the properties of the exponential map. Consider operators on w of the form $L=\sum_{s=0}^{n}\alpha_s P^s$, where $\alpha_s \in w$ and where P is the automorphism of w induced by raising to the pth power in $[k]$. Define

$|L|=\max\{|a|_v\}$. The author proves that a group a with conductor π^{e+1} is analytic if and only if there is an L with $|L|=1$ such that $E(La, \pi^i) \in a$ for all $a \in \mathfrak{v}$ and $0<i\leqq c$, and if that is the case there exist operators L_{ij}, $0<i\leqq j\leqq c$, with $|L_{ii}|=1$ such that the elements of the form $\prod_{j=1}^{c} E(L_{ij}a, \pi^j)$ for $a \in \mathfrak{v}$ and $0<i\leqq c$ generate $a\cap(1+\pi\mathfrak{v})$ modulo $1+\pi^{e+1}\mathfrak{v}$. *J. Tate.*

Whaples, G. Generalized local class field theory. IV. Cardinalities. Duke Math. J. **21** (1954), 583–586.

With the same notations as in the preceding review, the ith image group of a multiplicative group a is the additive group of the residues $[\xi]$ of integers ξ such that $1+\xi\pi^i \in a$. If each of these image groups is open, i.e. is the group of zeros of some additive polynomial, then a is called pseudo-analytic. Assume that the cardinal number of $[k]$ is $\aleph \geqq \aleph_0$. Then the number of analytic groups of finite index is $\aleph$, while the number of pseudo-analytic groups of finite index is $2^{\aleph}$. Hence pseudo-analyticity does not imply analyticity. *J. Tate* (Cambridge, Mass.).

Whaples, G. Additive polynomials. Duke Math. J. **21** (1954), 55–65.

Let k be a field of characteristic p. $f(x) \in k[x]$ is called an additive polynomial (a.p.) if $f(x+y)=f(x)+f(y)$ holds. An a.p. is of the form $\sum \alpha_\nu x^{p^\nu}$. A.p.'s have been studied by O. Ore [Trans. Amer. Math. Soc. **35** (1933), 559–584; **36** (1934), 243–274; Ann. of Math. (2) **34** (1933), 480–508], H. Hasse and E. Witt [Monatsh. Math. Phys. **43** (1936), 477–492] and E. Artin [Studies and essays presented to R. Courant, Interscience, New York, 1948, pp. 1–5; MR **9**, 266]. Here the author assumes that k has no inseparable extension, and k has for each integer n at most one extension of degree n and has exactly one extension of degree p. These axioms are assumed for the residue class field of a local field K, for which the generalized class-field theory can be established. Now in the additive group k^+ a subgroup $\mathfrak{v}$ is defined to be open when there is a polynomial $u(x) \in k[x]$ such that $\mathfrak{v}\supset u(k)$. Then, by a suitable a.p. $f(x)$, $\mathfrak{v}=f(k^+)$ holds, and $(k^+:f(k^+))$ equals the number of zeros of $f(x)$ in k. Moreover, there is an a.p. $g(x)$ for $\mathfrak{v}$ such that $\mathfrak{v}=g(k^+)$ and every a.p. $f(x)$ with $f(k^+)\subset\mathfrak{v}$ is of form $g(h(x))$. There are many corollaries of these results. Finally, with respect to the linear topology in k^+ as introduced above the continuous endomorphisms of k^+ defined by a.p.'s are everywhere dense in the ring of all continuous endomorphisms of k^+. *Y. Kawada.*

Crampton, T. H. M., and Whaples, G. Additive polynomials. II. Trans. Amer. Math. Soc. **78** (1955), 239–252.

Let k be a field of characteristic p and have no inseparable extension. Let $\{h_\nu(x)\}$ be a finite sequence of elements of $k[x]$ and define the linear transformation (l.t.) S of $k[x]^+$ into itself by $S\phi(x)=\sum_\nu \phi(h_\nu(x))$ $(\phi(x) \in k[x])$. Let $\mathbf{S}$ be the set of all such S's. Then $\mathbf{S}$ is a ring under addition and composition, and $Sf(\phi(x))=f(S\phi(x))$ holds for every additive polynomial (a.p.) $f(x)$. Then the main result concerning $\mathbf{S}$ is that $\mathbf{S}$ is everywhere dense in the ring E' of all l.t. of $k[x]^+$ into itself which commute with l.t. of $k[x]^+$ induced by the a.p. Here E' is topologized by taking as neighborhood of 0 the set of l.t. which annihilate some finite-dimensional subspace. This result together with the results on a.p. [see the preceding review] is applied to the proof of the existence theorem in the generalized class-field theory [see the paper reviewed fourth above]. The main purpose of this paper is to determine, for given $u_1(x) \in k[x]$, all decompositions of the forms $u_1(x)=f_1(u_2(x))$, $u_1(x)=f_1(u_2(x))+f_2(x)$, $u_1(x)=u_2(f_1(x))$, $u_1(x)=u_2(f_1(x))+f_2(x)$, $u_1(x)=f_1(u_2(f_2(x)))+f_3(x)$, where the $u_2(x)$ are arbitrary polynomials and $f_i(x)$ are a.p. For each $u_1(x) \in k[x]$ there is a maximal decomposition of each of these five types, of which all other decompositions of the same types are consequences. *Y. Kawada* (Tokyo).

See also: Séminaire P. Dubreil, p. 451; Flanders, p. 454; Sagastume Berra, p. 458; Rogers, p. 465.

Geometry of Numbers, Diophantine Approximation

Rogers, K. On the generators of an ideal, with an application to the geometry of numbers in unitary space U_2. Amer. J. Math. **77** (1955), 621–627.

The author defines an n-dimensional lattice in the space of n complex co-ordinates, the integers of some complex quadratic field $\mathfrak{K}$ taking on the rôle of the rational integers in the ordinary case. For $n=2$ he shows that the analogue of Mahler's compactness theorem [Proc. Roy. Soc. London. Ser. A. **187** (1946), 151–187; MR **8**, 195] holds, and so the analogue of Mahler's existence theorem for critical lattices may be deduced. It is stated that proofs for general n will be published jointly with Swinnerton-Dyer. An application is given in the paper reviewed below. The proofs depend on two results valid for any algebraic number field $\mathfrak{K}$. (I) If p, q, r, s are integers (of $\mathfrak{K}$) and the greatest common ideal divisors of p, q and r, s are equal, then to every integer u of $\mathfrak{K}$ there is a 2×2 integer matrix A such that $(p, q)A=(r, s)$. The case $u=1$ is Satz 35 of Hilbert's Zahlbericht. (II) For any rational integer $e>0$ there is a finite set of integral 2×2 matrices U_m such that every integral 2×2 matrix A with $|\text{norm}(\det A)|\leqq e$ is of the type $A=U_mT$, where T and T^{-1} are both integral. *J. W. S. Cassels.*

Rogers, K. Complex homogeneous linear forms. Proc. Cambridge Philos. Soc. **52** (1956), 35–38.

Let $\omega=\frac{1}{2}(-1+i\sqrt{7})$; then 1 and ω form a basis for the ring $Z(\omega)$ of integers in the quadratic field $Q(i\sqrt{7})$. Using results of the author and H. P. F. Swinnerton-Dyer (not yet published), the following theorem is proved: Let $\xi=ax+by$, $\eta=cx+dy$, where a, b, c, d are complex numbers and $ad-bc=\Delta\neq0$. Then there exist $x, y \in Z(\omega)$, not both zero, such that

$$\max(|\xi|, |\omega|)\leqq\left(\frac{2|\Delta|}{21^{\frac{1}{2}}-3}\right)=\mu.$$

The sign of equality is necessary only for forms which, apart form unimodular factors, can be reduced by transformations with coefficients in $Z(\omega)$ and determinant ±1 to $\mu\xi_0$, $\mu\eta_0$, where

$$\xi_0=x-\frac{1+i\sqrt{3}}{2}y,$$

$$\eta_0=\frac{3-\sqrt{21}+i(\sqrt{7}-\sqrt{3})}{4}x+\frac{\sqrt{3}-\sqrt{7}}{2}iy.$$

W. J. LeVeque (Ann Arbor, Mich.).

Chalk, J. H. H. On the product of n homogeneous linear forms. Proc. London Math. Soc. (3) **5** (1955), 449–473.

Let Λ be a lattice of determinant Δ in n-dimensional Euclidean space. The cases $n=3$, 4 are proved of the following conjecture: there exist n lattice points gener-

ating Λ and lying in the region $|x_1\cdots x_n|\leq 2^{-n+1}\Delta$. Also, the situation when not all are interior points is determined. The case $n=2$ was proved by H. Minkowski [Math. Ann. **54** (1900), 91–124] and generalized by the author [Proc. Cambridge Philos. Soc. (3) **49** (1953), 413–420; MR **14**, 1065]. The proof uses the results of Remak and Dyson that there exists an ellipsoid $p_1x_1^2+\cdots+p_nx_n^2=1$ having n independent points of Λ on its boundary and no point of Λ except 0 inside. For $n=4$ the proof is short but for $n=3$ it involves a lengthy examination of many possibilities to find which of the points on the ellipsoid and simple combinations of them are effective. *L. Tornheim.*

Šidlovskiĭ, A. B. On transcendental numbers of certain classes. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 977–980. (Russian)

This is a continuation of the author's recent work on E-functions [same Dokl. (N.S.) **96** (1954), 697–700; **100** (1955), 221–224; MR **16**, 117, 907]. The new results are similar to those announced earlier. *W. J. LeVeque.*

Turowicz, A. Sur une propriété des nombres irrationnels. Ann. Polon. Math. **2** (1955), 103–105.

It is shown that player A can always force the series $\sum_1^\infty a_n$ to converge to an irrational number, if he and B alternately choose the successive terms in such a way that $\{a_n\}$ is a decreasing sequence of positive real numbers. The proof depends on the Cantor expansion $\sum_1^\infty c_n/n!$ (c_n integers, $0\leq c_n<n$) of a real number.

W. J. LeVeque (Ann Arbor, Mich.).

Kurzweil, J. On the metric theory of inhomogeneous diophantine approximations. Studia Math. **15** (1955), 84–112.

Let ϕ be the homomorphism mapping the additive group of reals onto a circle K of unit circumference, such that for x real, $x'=\phi(x)=\{(2\pi)^{-1}\cos 2\pi x, (2\pi)^{-1}\sin 2\pi x)\}$. Lebesgue measure μ is defined on K in an obvious way. If g and h are real, $g<h$, then the interval $I[g, h]$ is the set of points x', where $g\leq x\leq h$. Let B denote a non-empty set of non-increasing sequences $\{b_k\}$ of positive numbers such that $\sum b_k$ diverges. A number $x\in(0, 1)$ belongs to the set $\alpha(B)$ if for every $\{b_k\}\in B$, almost every $\eta\in K$ belongs to an infinite number of intervals $I[kx-b_k, kx+b_k]$ ($k=1, 2, \cdots$). Let $\tilde{B}$ be the union of all sets B. It is shown that $\alpha(\tilde{B})$ coincides with the set Y of numbers $y\in(0, 1)$ for which there is a constant $c=c(y)>0$ such that $|y-p/q|>c/q^2$ for all sufficiently large q. This settles a question raised by Steinhaus (whether $\alpha(\tilde{B})$ contains all irrational numbers) since Y is of measure 0. The theorem is generalized (e.g., to the case of s linear forms in n variables), and the complementary theorem is proved that if B consists of a single sequence, then $\mu\alpha(B)=1$.

W. J. LeVeque (Ann Arbor, Mich.).

Evans, Arwel. A note on continued fractions and dimension theory. J. Reine Angew. Math. **195** (1956), 102–107 (1955).

If r, s are integers,

$$\varrho=r+\frac{1}{r+}\cdots,\quad \sigma=s+\frac{1}{s+}\cdots,$$

$K(a_1, a_2, \cdots, a_n)$ is the denominator of the continued fraction

$$\frac{1}{a_1+}\cdots\frac{1}{a_n}$$

and if u of the a_i are equal to r, the remaining $v=n-u$ of the a_i are equal to s, then it is shown that $A_1r^us^v\leq K\leq A_2\varrho^u\sigma^v$. It is deduced that the set $E(r, s)$ of real numbers whose continued-fraction representations have only r and s as partial quotients has Hausdorff dimension d lying between the roots of the equations

$$r^{-2x}+s^{-2x}=1,\quad \varrho^{-2x}+\sigma^{-2x}=1.$$

H. D. Ursell (Leeds).

See also: Séminaire P. Dubreil, p. 451.

ANALYSIS

Specht, Wilhelm. Abschätzung der Wurzeln algebraischer Gleichungen. III. Math. Z. **63** (1955), 324–330.

For a given polynomial of degree n

$$f(x)=x^n+a_1x^{n-1}+\cdots+a_n$$

let the zeros x_k be arranged in the order of decreasing modulus; let

$$P_k=|x_1x_2\cdots x_n|,\quad A=\max|a_k|^{1/k},\quad B=\max|a_k|\quad (1\leq k\leq n).$$

The author proves the two inequalities

$$P_k\leq s_kA^k(1-K^{n+1})^{\frac{1}{2}},\quad P_k<t_k(B),$$

where $s_k=k^{-\frac{1}{2}k}(k+1)^{\frac{1}{2}(k+1)}$, $K=k/(k+1)$, and $t_k(B)$ is a certain function of k and B. These results are generalizations of earlier results of the author [Math. Z. **52** (1949), 310–321; MR **11**, 431] and are derived from his earlier result that $P_k\leq N$, where $N=(1+|a_1|^2+\cdots+|a_n|^2)^{\frac{1}{2}}$ by a number of simple substitutions. *M. Marden.*

Loewner, Charles. On totally positive matrices. Math. Z. **63** (1955), 338–340.

A matrix is called totally positive if all its minors are non-negative. Let s_n denote the class of non-singular totally positive matrices of order n. Then s_n is a semigroup containing the identity matrix I. Now a matrix Ω is called an infinitesimal element of s_n if there exists a differentiable matrix function $U(t)$ on $[0, t_0]$ to s_n with $U(0)=I$ such that $[dU(t)/dt]_{t=0}=\Omega$. Let σ_n denote the class of infinitesimal elements of s_n. Finally an element U_0 of s_n is said to be generated by infinitesimal elements if there is a sectionwise continuous function $\Omega(t)$, $0\leq t<\infty$, with range in σ_n such that the solution $U(t)$ of the differential equation $dU(t)/dt=\Omega(t)U(t)$, $U(0)=I$, takes on the value U_0 for some $t>0$. The principal result of this note is to show that every element of s_n can be so generated. This result can be considered as a group-theoretical interpretation of a theorem by A. M. Whitney [J. Analyse Math. **2** (1952), 88–92; MR **14**, 732] which, incidently, constitutes an essential part of the proof.

R. S. Phillips (Los Angeles, Calif.).

Malliavin, Paul. La quasi-analyticité généralisée sur un intervalle borné. Ann. Sci. Ecole Norm. Sup. (3) **72** (1955), 93–110.

[An abstract appeared in C.R. Acad. Sci. Paris **240** (1955), 41–42; MR **16**, 453.] The author's problem is to discover what can be said about a function if it belongs to a quasi-analytic class on a finite interval $(0, R)$ and if

a sequence of derivatives vanishes at 0. Whereas for an infinite interval the vanishing of a sufficiently dense sequence of derivatives implies that the function vanishes identically, for a finite interval all that can be deduced is that the function is analytic at 0. The given function f is supposed to belong to a Denjoy-Carleman quasi-analytic class defined by $|f^{(n)}(x)| \leq M_n$; it has $f^{(n)}(0)=0$ except for a sequence $\{n_k\}$ of integers, of density zero, having prescribed "density d with respect to $\{M_n\}$" (a concept introduced in the paper). Then f is the restriction of a function that is analytic in a disk of center 0 and radius depending only on R and d; if $\sum 1/n_k < \infty$, this radius is R. An inverse theorem shows in particular that the radius of analyticity can be less than R. The inequality used to prove the main theorem can be used for other purposes, for instance to compare the growth in disks with the growth along a ray for an entire function defined by a lacunary series.

R. P. Boas, Jr. (Evanston, Ill.).

★**Albrecht, Rudolf, und Hochmuth, Hans. Übungsaufgaben zur höheren Mathematik. Teil III.** Verlag von R. Oldenbourg, München, 1956. 128 pp. DM 9.80.

For Teil II see MR 17, 19. This part contains problems on ordinary differential equations and partial differential equations of first order.

Theory of Sets, Functions of Real Variables

★**Ljapunow, A. A., Stschegolkow, E. A., und Arsenin, W. J. Arbeiten zur deskriptiven Mengenlehre.** Deutscher Verlag der Wissenschaften, Berlin, 1955. iii+108 pp. DM 15.15.

Translation of a collection of papers on the descriptive theory of sets: A. A. Lyapunov, Introduction; E. A. Šegol'kov, Elements of the theory of B-sets; V. Ya. Arsenin and A. A. Lyapunov, The theory of A-sets; A. A. Lyapunov, B-functions [Uspehi Mat. Nauk (N.S.) 5 (1950), no. 5(39), 11–13, 14–44, 45–108, 109–119; MR 12, 597].

Ribeiro de Albuquerque, José. Theory of projective sets. I, II. Univ. Lisboa. Revista Fac. Ci. A. (2) 1 (1951), 345–400; 2 (1952), 5–44. (Portuguese. French summary)

La prima parte di questo lavoro è dedicata a una rielaborazione e a una schematizzazione della teoria degli insiemi „analitici" che vengono generati mediante l'operazione (A) di Suslin a partire da una famiglia iniziale. Una novità consiste nello sviluppo della teoria indipendentemente da ogni topologia dello spazio in cui si considerano gli insiemi. Si ritrovano e vengono generalizzate così tutte le proprietà formali (proprietà di accessibilità, separabilità, ecc.) della teoria degli insiemi di Borel sviluppate da Lusin nel volume „Leçons sur les ensembles analytiques" [Gauthier-Villars, Paris, 1930], e viene generalizzata ai σ-corpi B_α la classificazione di Baire-de La Vallée Poussin. Fra i teoremi stabiliti dall'A. nella seconda parte del lavoro ve n'è uno che contiene il classico risultato che la proiezione di un insieme di Borel è un insieme analitico. L'A. stabilisce poi anche per gli insiemi proiettivi di classe superiore alcuni teoremi importanti che sussistono per gli insiemi di Borel e analitici. Vengono poi costruiti gli insiemi universali col metodo di Lavrentieff. Il lavoro termina con la risoluzione di alcuni problemi posti da Lusin nel suo volume citato sopra.

L. Giuliano (Zbl 49, 40).

Kurepa, Georges. Sur les correspondances multivoques. C. R. Acad. Sci. Paris 237 (1953), 1133–1135.

Soient r, s deux cardinaux, A, B deux ensembles de puissances α, β. Une correspondance entre A et B est dite (r, s) si elle associe, à chaque élément de A, r éléments de B, et à chaque élément de B, s éléments de A. On définit de même la correspondance $(\geq r, \leq s)$; l'A. démontre que s'il existe entre A et B une telle correspondance, on a $r\alpha \geq s\beta$. Il en déduit l'extension suivante du théorème de Cantor-Bernstein: s'il existe une correspondance (r, s) entre A et une partie de B et une correspondance (r, s) entre B et une partie de A, alors $\alpha=\beta$. Comme conséquence de la première proposition ci-dessus appliquée au cas fini, il énonce sans démonstration une proposition relative au nombre des „faces" d'un simplexe qui vérifient une certaine propriété. *R. de Possel* (Zbl 51, 40).

Sierpinski, W. Remarques sur un théorème de M. G. Kurepa concernant les correspondances multivoques. Matematiche, Catania 8 (1953), no. 2, 79–81.

A propos d'un lemme et d'un théorème du rapporteur [voir la note analysée ci-dessus] l'A. observe qu'il ne sait pas les démontrer sans faire usage de l'axiome de choix Z. Ainsi, par exemple, d'une part L_{21} (L_{rs}: s'il existe une (r, s)-correspondance de A sur B, alors $r\overline{A}=s\overline{B}$) entraîne P (la réunion d'une famille dénombrable d'ensembles à deux points est dénombrable) et d'autre part, on ne sait pas prouver P sans supposer Z. De même, l'A. remarque que sans supposer Z il ne sait pas prouver déjà T_{22} (T_{rs}: s'il y a une (r, s)-transformation de A en B et de B en A, alors $\overline{A}=\overline{B}$). *G. Kurepa* (Zbl 52, 285).

Schmidt, Jürgen. Eine verallgemeinerte Wohlordnung und die Endlichkeitsbedingungen der Ordnungstheorie. Arch. Math. 6 (1955), 374–381.

Sei M eine geordnete Menge. Als die allgemeinste Verallgemeinerung der wohlgeordneten Mengen [w.M.] sind wohl die $\min(E)$ zu deuten (dieselbe habe ich „teilweise wohlgeordnet", dann „wohlgeordnet" und schliesslich „rangiert" genannt; ranged sets; ensembles rangés). Es handelt sich da um geordnete Mengen, in denen jede Kette wohlgeordnet ist. Eine wichtige Klasse derselben sind die Bäume oder verzweigte Tafeln zu nennen.

M ist finitär, falls es eine endliche Untermenge A gibt so, dass für jedes $x \in M$ die Relation $x \geq x'$ mit $x' \in A$ stattfindet. Higman [Proc. London Math. Soc. (3) 2 (1952), 326–336; MR 14, 238] betrachtet die „finite basis property" (Fbp) (E) (jede Untermenge ist finitär) als eine Verallgemeinerung der w.M. Wenn jedes $x \in M$ als Supremum endlich vieler Elemente einer Fbp-Untermenge darstellbar ist, so ist auch die ganze Menge M eine Fbp-Menge [cf. Pickert, Math. Ann. 121 (1949), 100–102; MR 11, 5]. Unterwegs werden auch breitenendliche bzw. längenendliche M betrachtet; das sind die M, in denen jede Anti-Kette bzw. jede Kette endlich ist.

G. Kurepa (Zagreb).

Sce, M. Sul postulato della continuità della retta e su alcune sue applicazioni nella geometria elementare. Period. Mat. (4) 33 (1955), 215–225.

Scorza Toso, Annamaria. Sugli estremi parziali di una funzione di due variabili e la nozione di semicontinuità in una famiglia di insiemi. Rend. Sem. Mat. Univ. Padova **24** (1955), 93–102.

Let A be a set in the xy-plane, A_y its projection on the y-axis, and let $S(y_0)$ be the intersection of A with the line $y=y_0$. For a function $f(x, y)$ defined on A, let $\epsilon'(y)$, $\epsilon''(y)$ be the inf and sup of f on $S(y)$. According to G. Scorza-Dragoni [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **6** (1927), 579–583; **11** (1930), 865–872], A has the "property c)" if every subset of A which is closed in A, has a projection on the y-axis, which is closed in A_y. Similarly he defines the "property a)" by means of open sets. In the present paper relations between the semi-continuity of $f(x, y)$ and of $\epsilon'(y)$, $\epsilon''(y)$ are discussed in connection with the properties a) and c).

A. Rosenthal (Lafayette, Ind.).

Badell Portuondo, Enrique. Partial derivatives of unitary vectors in orthogonal curvilinear coordinates. Rev. Soc. Cubana Ci. Fis. Mat. **3** (1955), 103–108. (Spanish)

Simboan, G. Dérivées directes d'ordre p, des fonctions à valeurs en espaces de Banach. Com. Acad. R. P. Romîne **5** (1955), 1139–1144. (Romanian. Russian and French summaries)

The author extends to functions with values in a Banach space B theorems of Alexiewicz [Studia Math. **11** (1950), 185–196; MR **12**, 507] and Dieudonné [Ann. Sci. Ecole Norm. Sup. (3) **61** (1944), 231–248; MR **7**, 246]. If x is a function defined on a real interval I with values in B, let

$$\Delta^p x(t_0; h_1, \cdots, h_p)=\Delta^{p-1}x(t_0+h_p; h_1, \cdots, h_{p-1})-\Delta^{p-1}x(t_0; h_1, \cdots, h_{p-1});$$

the direct derivative of x at t_0 of order p is the limit of the ratio $\Delta^p x(t_0; h_1, \cdots, h_p)/h_1\cdots h_p$ as all h_i tend to zero. Alexiewicz's theorem concerns approximate derivatives. Dieudonné's theorem gives conditions necessary and sufficient for existence of the direct derivative of order p; for this generalization to hold the space must be of type (D) [Pettis, Duke Math. J. **5** (1939), 254–269], that is, each Lipschitzian function from I into B is differentiable almost everywhere. *M. M. Day* (Urbana, Ill.).

Nikodým, Otton Martin. Sur l'extension d'une mesure non Archimédienne, simplement additive sur une tribu de Boole simplement additive, à une autre tribu plus étendue. I. Les "bouts" dans une chaîne. II. "Agrégats" et leur norme. C. R. Acad. Sci. Paris **241** (1955), 1439–1440, 1544–1545.

Both notes consist of definitions preparatory to the topic stated in the title (which is to be presented later on). In particular, the first note defines extensions of the concepts of inf and sup of sets of real numbers to subsets of a quasi-ordered chain. By means of them, in the second note the usual definitions of both the absolute variation μ^{-*} of a finitely additive set function μ and of the upper integral of a real-valued function X with respect to $\bar{\mu}^*$ are restated in the following setting: The domain of μ is a Boolean subalgebra B of another such algebra $B'\neq B$ with the same 0 and 1; its range is a linearly ordered field ϕ; X is a simple function on B' with values in ϕ.

H. M. Schaerf (St. Louis, Mo.).

Nicolescu, Lilly-Jeanne. Sur une extension du critère de compacité d'Arzèla. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 545–552. (Romanian. Russian and French summaries)

A real-valued function f on $I=[0, 1]\times[0, 1]$ is said to be hyperbolically (or h-) continuous provided for each $(x, y)\in I$ and each $\varepsilon>0$ there exists $\delta>0$ such that $|f(x', y')-f(x', y)-f(x, y')+f(x, y)|<\varepsilon$ for all $(x', y')\in I$ with $|x-x'|<\delta$, $|y-y'|<\delta$. The space C_H of all such functions is topologized by means of the pseudonorm $\|f\|=\sup_{(x,y)\in I}|f(x, y)-f(x, 0)-f(0, y)|$, and is shown to be complete. The properties of h-equicontinuity, h-equiboundedness, and h-equiderivability are defined. It is proved that for a subset of C_H to be compact, the first two of these properties are necessary and the three together are sufficient. [Reviewer's notes: 1) In the reference to Bögel, the date should be 1933. 2) The equivalence relation in § 2 appears to be defined incorrectly, as two equivalent functions f and g need not have $\|f-g\|=0$.]

V. L. Klee (Los Angeles, Calif.).

Kudryavcev, L. D. On differentiable mappings. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 12–14. (Russian)

In what follows f is a differentiable mapping into R^n of a domain $G\subset R^n$, B, B^* are the boundaries of G, $f(G)$, and E, E_a denote, respectively, the subsets of G at which f has Jacobian zero and at which f takes the value $A\in f(G)$. The author establishes the following analogues of two propositions from the theory of complex functions: I. If E has no limit point neither E_a. II. The hypotheses: G simply-connected, $f(B)=B^*$, $\mathrm{Int}(E)=0$ and f oriented and locally monotone, imply that f is monotone.

L. C. Young (Madison, Wis.).

See also: Marczewski, p. 468; Zaslavskiĭ, p. 517; Hsu, p. 517.

Theory of Measure and Integration

Leĭfman, L. Ya. On passage to the limit under the Lebesgue integral sign. Ukrain. Mat. Ž. **7** (1955), 134–141. (Russian)

The author considers Lebesgue spaces [V. A. Rohlin, Mat. Sb. N.S. **25(67)** (1949), 107–150; MR **11**, 18] of finite measure and defines isomorphism mod zero of two functions f and g to mean the existence of a measurability-and-measure-preserving T defined almost everywhere on the space X such that $f(Tx)=g(x)$ everywhere T is defined. Lebesgue's dominated convergence theorem is generalized by proving that when f_n converges in measure to zero a sufficient condition that $\int_E f_n d\mu=0$ for every measurable E is that there exists a sequence of functions g_n with a summable majorant such that, for each n, g_n is isomorphic mod zero to f_n. The author gives conditions equivalent to this hypothesis; these conditions take different forms when the measure is continuous and when it is discrete. *M. M. Day* (Urbana, Ill.).

Marczewski, E. On translations of sets and a theorem of Steinhaus. Prace Mat. **1** (1955), 256–263. (Polish. Russian and English summaries)

A systematic survey with proofs of a proposition announced earlier [Colloq. Math. **2** (1951), 309], connected with the well-known theorem by Steinhaus [Fund. Math. **1** (1920), 93–104; cf. Hadwiger, MR **8**, 255; Kestelman,

MR 9, 188, 274; Eggleston, MR 11, 166]. The considerations take place in R^n. Each L-measurable set of measure >0 has (S_n)-property for $n=1, 2, \cdots$ (Th. 1). If CE is of measure 0 or of first cathegory, then $E \in (T)$ (Th. 3). Problems: Does there exist a perfect set of measure 0 having the property (S_3)? Does there exist a set F_σ of measure 0 with the (T)-property? Definitions: A set E has the (S_n)-property, if there exists a real $\eta>0$ such that for every A of diameter $<\eta$ and $\overline{A} \leqq n$ there exists a translation of A into E. The set E has the (T)-property, if every set of $\leqq \aleph_0$ points admits a translation into E.

G. Kurepa (Zagreb).

Sikorski, R. On the determination of measure by a function of an elementary figure. Prace Mat. **1** (1955), 285–291. (Polish. Russian and English summaries)

The following theorem is well-known: For each non-negative additive function ϕ of an elementary figure in R_k there exists a Borel measure μ whose value on each closed interval P_0 is the greatest lower bound of the values of ϕ on closed intervals containing P_0 in their interior. The author proves it anew by mapping R_k into an auxiliary topological space R^* so that ϕ induces a content on a ring K of compact subsets of R^*. Then the measure on the σ-ring generated by K to which this content can be extended induces μ.

H. M. Schaerf.

Haupt, Otto, und Pauc, Christian Y. Bemerkungen über Inhalte und Masse in lokal bikompakten Räumen. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1955**, 187–218.

The authors discuss contents (non-negative and finitely additive set functions defined on a Boolean ring of sets) in locally compact spaces (with no separation axioms assumed). A content j is called adapted to the topology of a space R if (1) every set in the domain of j has compact closure, (2) j is finite, (3) the boundary of each set in the domain of j is also in the domain of j, and each such boundary can be included in an open set U in the domain of j with $j(U)$ arbitrarily small, and (4) the open sets in the domain of j constitute a basis for the topology of R. The main purpose of the paper is to show that every adapted content can be extended to a Borel measure. The relations among the authors' methods and results and those of Radon, Bourbaki, and the reviewer are discussed with some care. The authors remark that their constructions depend on sets alone (and not on the points of R), so that those constructions extend immediately to the purely lattice-theoretic concept of a topological space. The paper concludes with an appendix that contains the proofs of the pertinent lattice-theoretic lemmas.

P. R. Halmos (Chicago, Ill.).

Morse, Marston, and Transue, William. Semi-normed vector spaces with duals of integral type. J. Analyse Math. **4** (1954/55), 149–186.

The authors state that the present paper, and others to appear shortly, are destined to extend their earlier work [see, e.g., Ann. of Math. (2) **51** (1950), 576–614; MR **12**, 110] about bilinear forms on the product of function spaces from the case in which the domain of the functions is a subset of Euclidean space to that in which the domain is a general locally compact space E.

Part I is accordingly devoted to a detailed examination of what one may term complex Radon measures on E, here referred to as C-measures on E. The method used is that suggested by Bourbaki's treatise [Intégration, Chap. I, II, III, IV, Actualités Sci. Ind., no. 1175, Hermann, Paris, 1952; MR **14**, 960]. In brief, a C-measure is a complex-linear form α on the vector space $\mathcal{K}_C$ of all continuous complex-valued functions on E having compact supports, the form α being subject to a certain continuity condition exactly analogous to that imposed for real Radon measures [Bourbaki, op. cit., p. 50]. Having established (Theorem 3.1) the existence for each C-measure α of a corresponding positive measure $|\alpha|$, the "total variation" of α, the set of x in C^E for which $\int^* |x|\, d|\alpha|$ is finite is a vector space $\mathcal{T}_C{}^1(\alpha)$ on which $\int^* |x|\, d|\alpha|$ acts as a seminorm. The closure of $\mathcal{K}_C$ in $\mathcal{T}_C{}^1(\alpha)$ is the set $\mathcal{L}_C{}^1(\alpha)$ of functions integrable for α. It turns out that, if one introduces the real Radon measures $\mu=(\mathrm{Re}\,\alpha)|\mathcal{K}$ and $\nu=(\mathrm{Im}\,\alpha)|\mathcal{K}$ and the Riesz decompositions of these, so that $\alpha=\mu^+-\mu^-+i\nu^+-i\nu^-$, then integrability of x for α is equivalent to that of each of the Riesz components of x for each of the positive measures $\mu^+, \mu^-, \nu^+, \nu^-$, this last in the sense of Bourbaki.

Part II introduces some special types of vector subspaces A of C^E. The first requirement is that A be equipped with a seminorm $\mathcal{M}^A$ which is to be monotone in the sense that $|x|\leqq|y|$ shall imply $\mathcal{M}^A(x)\leqq\mathcal{M}^A(y)$; it is also assumed that $\mathcal{M}^A$ is not identically 0. The so-called MT-spaces are characterised by two further conditions. Condition I requires further that $\mathcal{K}_C$ be dense in A relative to $\mathcal{M}^A$, and that A be stable under complex conjugation. From this it follows that any element α of the dual A' of A is such that $\hat{\alpha}=\alpha|\mathcal{K}_C$ is a C-measure. Condition II demands that each x in A be integrable for $\hat{\alpha}$, and that $\alpha(x)$ be equal to $\int x d\hat{\alpha}$. The map $\alpha\to\hat{\alpha}$ is then an algebraic isomorphism of A' onto a vector space $\mathcal{A}'$ of C-measures, but the implied identification is expressly avoided. A direct characterisation of $\mathcal{A}'$ is given in Theorem 10.1. By means of the Hahn-Banach theorem, it is shown that $\mathcal{M}^A$ admits an extension from A to C^E via the equation

$$\mathcal{M}^A(x)=\sup\{\int^* |x|\, d|\hat{\alpha}| : \hat{\alpha} \in \mathcal{A}',\ \|\hat{\alpha}\|_{\mathcal{A}'}\leqq 1\},$$

where

$$\|\hat{\alpha}\|_{\mathcal{A}'}=\sup\{|\alpha(u)| : u \in \mathcal{K}_C,\ \mathcal{M}^A(u)\leqq 1\}.$$

This extension is shown (Theorem 12.1) to be countably subadditive on C^E. At this stage one can mimic again Bourbaki's process by introducing the space $\mathcal{F}^A$ of maps $f \in C^E$ for which $\mathcal{M}^A(f)<+\infty$ and showing (Theorem 12.3) that the result is a complete seminormed space. If Ω^A is introduced as the intersection of the $\mathcal{L}_C{}^1(\hat{\alpha})$ as $\hat{\alpha}$ ranges over $\mathcal{A}'$, then the closure of A in $\mathcal{F}^A$ is contained in $\Omega^A\cap\mathcal{F}^A$.

Two MT-spaces A and B are said to be compatible if $\mathcal{A}'=\mathcal{B}'$ and $\mathcal{M}^A=\mathcal{M}^B$ on $\mathcal{K}_C$. It is shown (Theorem 13.1) that A (closure in $\mathcal{F}^A$) is the largest MT-space compatible with A, whilst the smallest is $\mathcal{K}_C$ itself.

The paper ends by discussing in detail two examples of MT-spaces, namely: (i) $B=\mathcal{L}_C{}^1(\beta)$, where $\beta\neq 0$ is a fixed C-measure, with $\mathcal{M}^B(f)=\int|f|\, d|\beta|$; (ii) $H=\mathcal{K}_C$, with $\mathcal{M}^H(f)=\sup\{|f(t)| : t \in E\}$.

R. E. Edwards (London).

Stoka, Marius. Sur la mesure de l'ensemble des cercles du plan. Gaz. Mat. Fiz. Ser. A. **7** (1955), 556–559. (Romanian. Russian and French summaries)

Kneser, Martin. Einige Bemerkungen über das Minkowskische Flächenmass. Arch. Math. **6** (1955), 382–390.

The author proves that, for any set A in 3-space R_3 which is the image under a Lipschitzian mapping of a

compact set in the plane, the 2-dimensional Minkowski measure of A equals the 2-dimensional Gross minimal measure of A. He mentions that the corresponding result holds for s-dimensional Minkowski measure in R_n for arbitrary n and $s<n$; but an example is given to show that this need not be true without the Lipschitz condition even for $s=2$, $n=3$. The following is also proved: The property of s-dimensional Minkowski measurability of a set A, as well as the s-dimensional Minkowski measure of A, does not depend on the dimension of the space R_n in which A is embedded. *W. H. Fleming.*

Cesari, Lamberto. L'area di Lebesgue come una misura. Rend. Mat. e Appl. (5) **14** (1955), 655–673.

Let T be a continuous transformation from a finitely connected closed Jordan region A in the plane into three-space [for definitions cf. L. Cesari, Surface area, Princeton, 1956]. Denote by $\mathfrak{M}$ the collection of all subsets of A, and by $\mathfrak{G}$, $\mathfrak{B}$, the collections of all sets in $\mathfrak{M}$ which are respectively open in A, Borel. For M in $\mathfrak{M}$ define $\psi(M, T)$ to be the greatest lower bound of the Lebesgue areas of the surfaces determined by restricting T to sets G in $\mathfrak{G}$ which contain M. If T is of bounded variation, it is shown that a necessary and sufficient condition in order that T be absolutely continuous is that ψ be continous according to Lebesgue measure and completely additive on $\mathfrak{B}$.

Denote by T_r for $r=1, 2, 3$ the plane transformation obtained by projecting T on each of the coordinate planes. Let $\mathfrak{M}_0$ be the collection of all sets in $\mathfrak{M}$ which are unions of maximal continua of constancy for T in A; and let $\mathfrak{G}_0$, $\mathfrak{B}_0$ denote the sets common to $\mathfrak{M}_0$ and $\mathfrak{G}$, $\mathfrak{B}$ respectively. If T is of bounded variation, then for each set in $\mathfrak{M}_0$ a vector function $d=(d_{*1}, d_{*2}, d_{*3})$ is constructed so that d is completely additive on $\mathfrak{B}_0$ and on every set G in $\mathfrak{G}_0$ the value of d_{*r} is the algebraic total variation of the flat surface obtained by restricting T_r to G for $r=1, 2, 3$.
P. V. Reichelderfer (Columbus, Ohio).

Federer, Herbert. An addition theorem for Lebesgue area. Proc. Amer. Math. Soc. **6** (1955), 911–914.

Let K be a finite set of compact topological cells of dimensions not exceeding two with disjoint interiors, with union X, and such that the boundary of each cell in K is the union of certain lower-dimensional cells in K. Consider a continuous map f of X into Euclidean n-space where n is at least two. If the one-dimensional Lebesgue area of $f|B$ is finite for every one-cell B in K, then it is shown that the two-dimensional Lebesgue area of f is the sum of the two-dimensional Lebesgue areas of $f|A$ over all the two-cells A in K. *P. V. Reichelderfer.*

Functions of a Complex Variable, Generalizations

★ **Behnke, Heinrich, und Sommer, Friedrich. Theorie der analytischen Funktionen einer komplexen Veränderlichen.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXVII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. x+582 pp. DM 66.00.

This book is the outgrowth of lectures given by the authors on the theory of functions of a complex variable over a number of years at the University of Münster. It pursues two objectives. The first is to give a rigorous detailed account of the classical theory of functions of a complex variable for beginning students who have had prior training in classical real analysis. The second is to give an account of the theory of Riemann surfaces and analytic functions on Riemann surfaces. Here the material includes developments of the last decade and is intended not only as an introduction but also as a reference for the specialist. The exposition is thoroughgoing and, with the exception of results form surface theory, which are intentionally stated without proof, results from other fields which are used are carefully developed. Discussion and comments illuminate the exposition. Material important for applications has been included. While no attempt is made at bibliographical completeness, the references made to texts and papers are copious and are definitely up-to-date.

Among the topics treated in the first half of the book mention should be made of the strong form of the Cauchy integral theorem. In the chapter on expansions the theory of orthogonal functions and the Bergman kernel function are given prominence. In the part on Riemann surface theory a fusion is sought of the Riemannian and Weierstrassian points of view. The development includes an account of the uniformization theorem, automorphic functions and abelian integrals. The treatment of the Riemann-Roch theorem is that due to Teichmüller [Deutsche Math. **6** (1941), 257–265; MR **8**, 327] which is based on uniformization theory and the use of the Poincaré theta series which assures the existence of quadratic differentials. The last chapter culminates in an account of the generalization of the Runge theorem to non-compact Riemann surfaces due to Behnke and Stein and the consequent existence of non-constant analytic functions on such surfaces.

The contents of the book follows. Chapter 1 treats elementary complex analysis, line integrals, sequences of functions. The central theorems of the Cauchy theory are developed in Chapter 2 which also includes a section on normal families of analytic functions and one on harmonic functions. Chapter 3 treats analytic continuation, singularities of analytic functions, the calculus of residues, the Mittag-Leffler theorem, the Runge theorem and orthogonal functions. Chapter 4 is an account of the classical theory of conformal mapping. Boundary behavior and the theorems of Bieberbach and Koebe are treated. In Chapter 5 after the basic concepts of Riemann surface theory are expounded, the uniformizationth eorem is established. Chapter 6 treats the theory of functions on Riemann surfaces and abelian differentials. The final paragraph of this chapter gives an account of "Schleifenintegrale" together with applications to the gamma, beta, and hypergeometric functions. *M. Heins.*

Myškis, A. D., and Gil', G. V. On a problem of N. N. Luzin. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1(63), 143–145. (Russian)

Among the unsolved problems left by the late N. N. Lusin [cf. Fëdorov, Uspehi Mat. Nauk (N.S.) **7** (1952), no. 2(48), 7–16; MR **13**, 810] is that of the structure of the set of monogenicity of a function $f(z)$ which is continuous in some domain D; the set of monogenicity $\mathfrak{M}_z$ of $f(z)$ at a given point z of D is defined as the intersection $\bigcap_{\varepsilon>0} \overline{M}_\varepsilon$, where M_ε is the set of all complex numbers

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} \text{ with } z+\Delta z \in D \text{ and } 0<|\Delta z|<\varepsilon.$$

It was shown by A. D. Taimanov [ibid. **8** (1953), no. 5(57), 169–171; MR **15**, 612] that $\mathfrak{M}_z$ can fill a circle. It is

shown in the present note that a necessary and sufficient condition that a given closed set $\mathfrak{M}$ in the plane can be a set of monogenicity of some continuous function at some point z in D is that $\mathfrak{M}$ be connected. *A. J. Lohwater.*

Pompeiu, D. Observations sur la représentation des functions analytiques. Com. Acad. R. P. Române 3 (1953), 247–248. (Romanian. Russian and French summaries)

The author plans to study representations of analytic functions by means of "interior elements" and by means of "boundary elements". From the all-too-brief sketch, the reviewer cannot say precisely what the author has in mind. *M. O. Reade* (Ann Arbor, Mich.).

Kudryavcev, L. D. On properties of harmonic mappings of plane regions. Mat. Sb. N.S. **36(78)** (1955), 201–208. (Russian)

Proofs of results announced in an earlier paper [Dokl. Akad. Nauk SSSR (N.S.) **92** (1953), 469–471; MR **15**, 612]. *A. J. Lohwater* (Helsinki).

Pontryagin, L. S. On the zeros of some elementary transcendental functions. Amer. Math. Soc. Transl. (2) **1** (1955), 95–110.

Translated from Izv. Akad. Nauk SSSR. Ser. Mat. 6 (1942), 115–134; MR **4**, 214.

Biernacki, M. On a certain lacunary power series. Prace Mat. **1** (1955), 264–271. (Polish. Russian and English summaries)

The author proves that the functions $f(x)=1+x^{1!}+x^{2!}+\cdots$ and $h(x)=\log\{1/(1-x)\}/\log\log\{1/(1-x)\}$ are asymptotically equal for $x\to 1-$ (i.e. $\lim_{x\to 1-} f(x)/h(x)=1$). He uses a theorem of Cesàro on power series,

$$\lim_{x\to 1-}\sum_n a_n x^n/\lim_{x\to 1-}\sum_n b_n x^n=\lim_n a_n/b_n$$

under certain conditions, and an inequality of Čebyšev,

$$n(a_1b_1+\cdots+a_nb_n)\geqq(a_1+\cdots+a_n)(b_1+\cdots+b_n)$$

for positive decreasing a_i and b_i. An intermediate result is $f(x)\sim g(x)=\sum_n x^n/n\log\log n$; and the asymptotic behaviour of $g(x)$ is determined by dividing the series into the parts $n<$ or $>1/(1-x)$. *K. Zeller.*

Ricci, Giovanni. Fluttuazione relativa e punti singolari delle serie di potenze. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 3–24.

Verfasser setzt seine Untersuchungen über Fabrysche Sätze fort [siehe Ann. Mat. Pura Appl. (4) **38** (1955), 1–31; MR **17**, 139]. Es geht dabei um Kriterien dafür, daß 1 ein singulärer Punkt der Funktion $\sum a_n z^n$ (Konvergenzradius 1) ist. Dazu betrachtet Verfasser Abschnitte I_h ($h=1, 2, \cdots$) der Potenzreihe, die durch $(1-\theta)n_h\leqq m\leqq(1+\theta)n_h$ (wo $0<\theta<1$, n_h ganz, $n_h\to\infty$) gegeben sind. Jedem Abschnitt wird ein Richtungsfaktor γ_h zugeordnet. In den I_h sollen die a_n nicht zu klein sein ($|\mathrm{Re}(a_{n_h}e^{-i\gamma_h})|^{1/n_h}\to 1$) und wenig schwanken. Letzteres definiert Verfasser mit Hilfe der Summen

$$S_h=\sum_{m\in I_h} C_{n_h,m}\mathrm{Re}(a_m e^{-i\gamma_h}) \text{ mit } C_{n,m}=\frac{n!n!}{m!(2n-m)!}.$$

Für eine geeignete Nullfolge α_h, deren Abfall das Wachstum von n_h nicht ganz kompensiert ($1+\varepsilon<\delta\alpha_h n_h$ für gewisses $\delta>0$, $\varepsilon>0$), soll die „fluttuazione relativa" $F_{\alpha_h}\{S_h\}<\alpha_h^{(2\theta+\theta)n_h n_h}$ sein. Für eine $(N+1)$-gliedrige Zahlenfolge $\{u_0, \cdots, u_N\}$ setzen wir dabei

$$F_\alpha\{u_n\}=(V_1\{u_n\}-V_\alpha\{u_n\})/V_1\{u_n\}$$

mit $V_1\{u_n\}=|u_1-u_0|+\cdots+|u_{N-1}-u_N|$ (Totalschwankung) und $V_\alpha\{u_n\}=\sup(|u_{h_1}-u_0|+|u_{h_2}-u_{h_1}|+\cdots+|u_N-u_{h_s}|)$, wobei zur sup-Bildung die h_i mit $0<h_1<\cdots<h_s<N$ und $S\leqq\alpha N$ zugelassen sind (V_α ist also die größte Schwankung gewisser Teilfolgen). Das Hauptergebnis (Teorema A) besagt nun: Kann man n_h, γ_h, θ, α_h, δ, ε gemäß obigen Forderungen bestimmen, so ist 1 singulärer Punkt. Teorema B bringt eine wesentlich speziellere Aussage; es wird im wesentlichen nur gefordert, daß für $a_n=|a_n|e^{i\varphi_n}$ die Beziehung

$$\sum_{m\in I_h}|\varphi_{m+1}-\varphi_m|=O(n_h)$$

gilt. Teorema C sagt aus, daß mit $\sum a_n z^n$ auch alle „benachbarten" Potenzreihen den Voraussetzungen von Teorema A genügen. *K. Zeller.*

Šaginyan, A. L. On the speed of polynomial approximation on continua. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 190–192; errata, **107** (1956), 8. (Russian)

Notations: Ω est l'intérieur d'une courbe simple rectifiable L; E est un compact dans Ω, de diamètre $2d$; la distance de L au disque de diamètre $2d$ recouvrant E est $2d\alpha$; $\varrho(s)$ est la distance à E du point d'abscisse curviligne s sur L; posons $K=\alpha\exp\{-\int_L(\varrho(s))^{-1}ds\}$. Voici les principaux résultats annoncés. Th. 1. Supposons connexe le complémentaire E_∞ de E, et soit $G(z)=G(z,\infty)$ la fonction de Green de E_∞; alors on a, sur L, $G(z)>K$. Th. 2. Si $f(z)$ est holomorphe dans Ω, continue et de module $\leqq 1$ dans $\bar{\Omega}$, il existe pour chaque n un polynôme $P_n(z)$ de degré $\leqq n$ tel que, sur E, $|f(z)-P_n(z)|\leqq C\exp[-Kn]$ où C ne dépend que de Ω et E. [Comparer à Mergelyan, mêmes Dokl. (N.S.) **91** (1953), 1271–1274; MR **15**, 524.] *J. P. Kahane* (Montpellier).

Rahman, Q. I. A note on entire functions (defined by Dirichlet's series) of perfectly regular growth. Quart. J. Math. Oxford Ser. (2) **6** (1955), 173–175.

For an entire function defined by a Dirichlet series, and of finite (Ritt) order, the author establishes several connections between the maximum modulus on vertical lines and the index of the maximum term; these resemble the relations between the maximum modulus and the central index for an entire function defined by a power series. *R. P. Boas, Jr.* (Evanston, Ill.).

Dalovitch, Voĭn. Sur l'existence des valeurs limites de la résultante des fonctions, appartenant à la classe H_δ, $0<\delta<1$, et encore de certaines autres classes de fonctions analytiques. C. R. Acad. Sci. Paris **241** (1955), 1441–1444.

The author asserts that, if $f(z)=\sum a_n z^n$ is in some class H^δ, $\delta>0$ (that is, if the integral of $|f(z)|^\delta$ over the circle $|z|=r$ remains bounded as $r\to 1$), and if the real part of $\sum b_n z^n=g(z)$ is a Poisson-Stieltjes integral, then $\sum a_n b_n z^n$ has boundary (non-tangential) values almost everywhere on $|z|=1$. The result is obviously not true (take, e.g., $f(z)=\sum nz^n$, $g(z)=\sum 2^{-n}z^{2^n}$) and the proof based on a misunderstanding (see formula (12) of the paper). *A. Zygmund* (Chicago, Ill.).

Bagemihl, F., and Seidel, W. Some remarks on boundary behavior of analytic and meromorphic functions. Nagoya Math. J. **9** (1955), 79–85.

Examples are constructed of functions which are

analytic or meromorphic in $|z|<1$ and have special boundary properties. The function

$$g(z)=\Big[\prod_{j=1}^{\infty}\Big\{1-\Big(\frac{z}{1-n_j^{-1}}\Big)^{n_j^j}\Big\}\Big]^{-1},$$

where $n_j=3^j$, which is meromorphic in $|z|<1$ is shown to have the property that for each θ $(0\leqq\theta<2\pi)$, $f(z)\to 0$ as $z\to e^{i\theta}$ along almost all chords of $|z|<1$ terminating in $e^{i\theta}$. By a theorem of K. Meier [Comment. Math. Helv. **24** (1950), 238–259; MR **12**, 490] it is concluded that there do not exist non-constant analytic functions in $|z|<1$ with this limiting behavior. It is also shown that there exist analytic functions in $|z|<1$ which tend to infinity along each member of a certain set of spirals approaching $|z|=1$ and exhausting $|z|<1$. It is also shown that such a set of spirals cannot be replaced by an exhausting set of Jordan arcs terminating in points of $|z|=1$. *M. Heins.*

Bagemihl, Frederick, et Seidel, Wladimir. Valeurs évitées, mais non asymptotiques, pour les fonctions holomorphes dans le cercle-unité. C. R. Acad. Sci. Paris **241** (1955), 1195–1196.

Let $h(z)$ be regular in the unit circle U: $|z|<1$. A complex value η, finite or infinite, is called an avoided value for $h(z)$, if η is omitted by $h(z)$ but every value different from η, belonging to a certain neighborhood of η, is taken by $h(z)$ infinitely often in U. The authors prove the following two theorems: (1) There exists a function $G(z)$, regular and bounded in U, for which 0 is an avoided value but not an asymptotic value. (2) There exists a function $G(z)$, regular and of unbounded characteristic in U, for which 0 is an avoided value but not an asymptotic value. These theorems give a negative answer to a question raised by E. F. Collingwood and M. L. Cartwright [Acta Math. **87** (1952), 83–146, p. 120; MR **14**, 260]. The reviewer remarks that Ohtsuka also gives a negative answer to the same question [Trans. Amer. Math. Soc. **78** (1955), 294–304; MR **16**, 686]. *K. Noshiro* (Nagoya).

Lehto, Olli. Boundary theorems for analytic functions. Ann. Acad. Sci. Fenn. Ser. A. I. no. **196** (1955), 8 pp.

Let $w=f(z)$ be meromorphic in $|z|<1$ and E the closure of its radial boundary values $w=\lim_{r\to 1} f(re^{i\varphi})$ $(0\leqq\varphi<2\pi)$. The author proves that if the set of values of $f(z)$ is everywhere dense in the w-plane, $f(z)$ can omit at most two values outside E, and that if the set of values is not everywhere dense, $f(z)$ either takes or omits all values belonging to an arbitrary domain outside E. Secondly, it is shown that, $w=f(z)$ being of bounded type in $|z|<1$, if the set of values is everywhere dense, $f(z)$ can omit at most one value outside E. In the proof an idea due to Lohwater [Duke Math. J. **19** (1952), 243–252; MR **14**, 34] is used. By this second result the question raised by the author on p. 45 (p. 43, which is quoted in the paper under review, should be p. 45) of his paper [Ann Acad. Sci. Fenn. Ser. A. I. no. **177** (1954); MR **16**, 688] is solved provided that boundary value is defined as radial limit, whereas if the problem concerns angular limits, the question remains still open. Finally, a question raised by the reviewer [Trans. Amer. Math. Soc. **78** (1955), 294–304; MR **16**, 686] is answered in the negative. We remark that a generalization of Lindelöf's theorem on angular limits which is attributed here to Collingwood and Cartwright [Acta Math. **87** (1952), 83–146; MR **14**, 260] was already proved by Gross [Math. Z. **3** (1919), 43–64] in a more general form. *M. Ohtsuka.*

Linden, C. N. Functions regular in the unit circle. Proc. Cambridge Philos. Soc. **52** (1956), 49–60.

Let $f(z)=f(re^{i\theta})=u(r,\theta)+v(r,\theta)$ be regular for $|z|<1$. Results generalizing some of M. L. Cartwright [Quart. J. Math. Oxford Ser. **4** (1933), 246–257], relating the behavior of $|f(z)|$ as $r\to 1$ to that of $u(r,\theta)$ as $r\to 1$, are given. We may assume $f(0)=0$. Examples of the results are as follows.

Theorem 1. Let $\chi(x)$ be a positive continuous function for which $\int_{x_0}^{\infty}[\xi\chi(\xi)]^{-1}d\xi$ is finite. Suppose also that there is a finite constant d such that $\chi(x)$ and $(\log x)^d[\chi(x)]^{-1}$ increase to infinity with x for $x>x_0$. Then if

$$u(r,\theta)<\frac{1}{1-r}\Big[\chi\Big(\frac{1}{1-r}\Big)\Big]^{-2} \quad (r_0<r<1),$$

there is a constant K which depends only on r_0 and the function $\chi(x)$ such that when $|z|\leqq r$, $|f(z)|<K(1-r)^{-1}$ $(0\leqq r<1)$.

Theorem 3. Let $\varrho(x)$ be real, continuous and differentiable on adjacent intervals for $x>x_0$, $\varrho(x)\to\varrho>1$ as $x\to\infty$, $\varrho'(x)x\log x\to 0$ as $x\to\infty$. Let $\sigma(1/x)=\varrho(x)$. Then if $u(r,\theta)<(1-r)^{-\sigma(1-r)}$ $(r_0\leqq r<1)$, there is a constant K depending only on r_0 and the function $\sigma(1-r)$ such that, if $|z|\leqq r$, $|f(z)|<K(1-r)^{-\sigma(1-r)}$ $(r_0\leqq r<1)$. *E. Reich.*

Călugăreanu, G. Sur les fonctions univalentes. II. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Şti. **5** (1954), 15–26. (Romanian. Russian and French summaries)

In this paper, the author obtains necessary and sufficient conditions that $f(z)=1/z+\sum_1^{\infty}a_nz^n$, analytic for $|z|>1$, should be univalent for $|z|>1$. His technique exploits the Bieberbach "area principle" and contains Bieberbach's result as a special case. The author's formulas are too complicated to be reproduced here. *M. O. Reade.*

Uluçay, Cengiz. On the Bloch-Landau constants. Comm. Fac. Sci. Univ. Ankara. Sér. A. **7** (1955), 233–252. (Turkish summary)

Let $(T_\omega)_1$, $(T_\omega)_2$ and $(T_\omega)_3$ be respectively an equilateral triangle, an isoceles right triangle, and a 30°–60°–90° triangle. Let $\mathfrak{L}$ be the class of all functions, normalized by $|f'(0)|=1$, which map conformally a zero-angled circular triangle (modular triangle) inscribed in the unit circle onto one of the three triangles $(T_\omega)_n$ $(n=1,2,3)$. The author proves that the only function which is extremal in the class $\mathfrak{L}$ for the Landau problem (i.e., finding the radius of the largest circle covered by the values of all $f\in L$) maps the modular equilateral triangle on $(T_\omega)_1$. The class B consists of normalized conformal mappings of a circular triangle orthogonal to and inside of the unit circle with interior angles of π/p, π/q, π/r (with $(1/p)+(1/q)+(1/r)<1$, and p, q, r positive integers) onto $(T_\omega)_n$, $n=1,2,3$. The only function which is extremal for the Bloch problem in the class $\mathfrak{B}$ (i.e., the problem of finding the radius of the largest circle covered univalently by the values of all $f\in\mathfrak{B}$) is the mapping of the equiangular circular triangle with interior angles equal to $\pi/6$ onto $(T_\omega)_1$. The latter is proved by means of a generalization of the Schwarz lemma. *G. Springer* (Lawrence, Kan.).

Suvorov, G. D. On the prime ends of a sequence of plane regions converging to a nucleus. Amer. Math. Soc. Transl. (2) **1** (1955), 67–93.

Translated from Mat. Sb. N.S. **33(75)** (1953), 73–100; MR **15**, 244.

Kleiner, W. Démonstration du théorème de Osgood-Carathéodory par la méthode des points extrémaux. Ann. Polon. Math. **2** (1955), 67–72.

The author gives a new proof of the classical theorem that if $w=f(z)$ maps the exterior of a Jordan curve C on the region $|w|>1$ then the mapping can be extended to a homeomorphism of the boundaries. The proof is based on the representation of $f(z)$ as $\lim_{n\to\infty} f_n(z)$,

$$\log f_n(z)=i\vartheta_n+\sum_{k=1}^{n}\log[(z-\eta_k^{(n)})(\eta_0^{(n)}-\eta_k^{(n)})],$$

where $\eta_0^{(n)}, \eta_1^{(n)}, \cdots, \eta_n^{(n)}$ are a system of $(n+1)$ extremal points of the type used in defining the transfinite diameter of C, arranged in such a manner that

$$\prod_{k=1}^{n}|\eta_0^{(n)}-\eta_k^{(n)}|\leqq\prod_{\substack{k=0\\k\neq i}}^{n}|\eta_i^{(n)}-\eta_k^{(n)}|.$$

E. Reich (Santa Monica, Calif.).

Künzi, Hans P. Entwicklung und Bedeutung der konformen Abbildung. Elem. Math. **11** (1956), 1–15.

Tsuji, Masatsugu. On the modulus of a ring domain. Comment. Math. Univ. St. Paul **4** (1955), 1–3.

Let D be a doubly-connected domain in the z-plane which is bounded by the continua C_1 and C_2, and let D_1 be the simply-connected domain which contains D and is bounded by C_1. The author shows that, if M denotes the Riemann modulus of D, $M=\inf_v I(v)$, where $I(v)=\int_{C_1}\int_{C_1} G(z,t)dv(z)dv(t)$, $G(z,t)$ is the Green's function of D_1, $dv\geqq 0$ and $\int_{C_1} dv=1$. As an application he derives the well-known result that if C_1 is the unit circumference and C_2 contains the points $z=r$, $z=R$ $(1<r<R<\infty)$, the modulus attains its largest value if C_2 is the linear segment connecting the two points. *Z. Nehari.*

Strebel, Kurt. On the maximal dilation of quasiconformal mappings. Proc. Amer. Math. Soc. **6** (1955), 903–909.

The author considers quasiconformal mappings [cf. Ahlfors, J. Analyse Math. **3** (1954), 1–58, 207–208; MR **16**, 348] and asks the following question: Given an open set G in the plane and a subset E of G, when can a quasiconformal mapping $w(z)$ defined on $G-E$ be extended to G and have the same maximal dilation on G as on $G-E$. If we know only that $w(z)$ is defined and quasiconformal on $G-E$, a necessary and sufficient condition is that every compact subset of E be a null set with respect to AD functions [cf. Ahlfors and Beurling, Acta Math. **83** (1950), 101–129; MR **12**, 171]. If in addition we know that $w(z)$ admits a topological extension from $G-E$ to E, then it is sufficient that E be of σ-finite linear measure. If we know further that $w(z)$ admits of a quasiconformal extension from $G-E$ to G, then a sufficient condition that the maximal dilation not be increased is that E have two-dimensional measure zero. *H. L. Royden.*

Čao, T. I. On entire algebraic functions with two branches. Mat. Sb. N.S. **37(79)** (1955), 573–576. (Russian)

The author considers entire algebroid functions $x(y)$ defined by a relation of the form $F(x,y)\equiv x^2+2a_1(y)x+a_2(y)=0$, where a_1 and a_2 are entire functions. It is shown that if $x(y)$ has a branch, say $x_1(y)$, analytic at infinity and taking the value α there, then either α is a limit point of the abscissae of the multiple points of $F=0$ or $-2a_1(y)-x_1(y)$ possesses α as an asymptotic value at infinity. The proof involves elementary considerations. *M. Heins.*

Heins, Maurice. Remarks on the elliptic case of the mapping theorem for simply-connected Riemann surfaces. Nagoya Math. J. **9** (1955), 17–20.

The author proves by means of the Perron method the following existence theorem for harmonic functions: Given a compact Riemann surface F and a point $p_0\in F$, there exists a single-valued function $V(p)$ on F, harmonic outside p_0 and with a representation $V[p(t)]=R[\alpha\cdot t^{-n}]+h(t)$ in the neighbourhood of p_0. Here, t is a local uniformizer at p_0, $p(0)=p_0$, $h(t)$ harmonic, n any natural number and $\alpha=0$. The possibility of such a proof was mentioned in an earlier paper [Ann. of Math. (2) **50** (1949), 686–690; MR **11**, 93]. *K. Strebel* (Fribourg).

Andreian Cazacu, Cabiria. Rapports entre les surfaces riemanniennes normalement exhaustibles et les surfaces riemanniennes algébriques qui se rapprochent de ∞: A_∞. Acad. R. P. Romîne. Bul. Şti. Secţ. Şci. Mat. Fiz. **7** (1955), 529–542. (Romanian. Russian and French summaries)

This paper studies the relation between Riemann surfaces normally exhaustible in the sense of Stoïlow [Compositio Math. **7** (1940), 428–435; MR **2**, 85] and Riemann surfaces of class A_∞ in the sense of Volkovyskiĭ [Trudy Mat. Inst. Steklov. **34** (1950); MR **14**, 156]. It is shown that (1) normally exhaustible Riemann surfaces of the first kind having an infinity of ramification points belong to the class A_∞, (2) the surfaces of class A_∞ are normally exhaustible or not according as the lacunary set reduces to a point or is empty, (3) the surfaces of class A_∞ are regularly exhaustible. *M. Heins* (Providence, R.I.).

Myrberg, Lauri. Différentielles méromorphes sur des surfaces de Riemann ouvertes. C. R. Acad. Sci. Paris **241** (1955), 1194–1195.

Let Φ be a meromorphic differential on an open Riemann surface S, and denote by $\{S_n\}$ an exhaustion of S. Consider the component φ_n of Φ on S_n that shares the periods and singularities of Φ on S_n and is orthogonal to all total differentials of the first kind on S_n. The linear class of differentials Φ such that

$$\limsup_{n\to\infty}\|\Phi-\varphi_n\|_{S_n}<\infty$$

for all $\{S_n\}$ is designated by F.

The author easily shows that every differential of the first kind with a finite norm belongs to F, and that every total differential of the first kind in F has a finite norm. He also observes that, if $S\in O_{AD}$, then the differentials in F are determined by their periods and singularities. These statements are modifications of the theory of meromorphic differentials of Ahlfors [Comment. Math. Helv. **24** (1950), 100–134; MR **12**, 90] and Bader [Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 243–300; MR **16**, 1012]. *L. Sario* (Los Angeles, Calif.).

Sce, Michele. Monogeneità e totale derivabilità nelle algebre reali e complesse. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16** (1954), 321–325.

L'A. fa seguito a due Note precedenti [gli stessi Rend. (8) **16** (1954), 30–35, 188–193; MR **16**, 786]; prova che nelle algebre commutative le funzioni $y(x)$ a jacobiano non nullo tali che $y(x)dx$ sia chiusa sono le funzioni totalmente derivabili e che, se l'algebra è inoltre solenoidale, la forma $y(x)dx$, se chiusa, è anche cochiusa; avanza una presunzione circa la validità di una certa formula inte-

grale; sviluppa alcune applicazioni alla risoluzione nel campo reale di certe equazioni alla derivate parziali.

G. Scorza Dragoni (Zbl 55, 310).

See also: Markuschewitsch, p. 445; Carathéodory, p. 446; Malliavin, p. 466; Kudryavcev, p. 468; Vermes, p. 475; Orts, p. 475; Taam, p. 483; Kreyszig, p. 492; Tillman, p. 510; Calabi, p. 517; Warschawski, p. 540.

Harmonic Functions, Potential Theory

Miles, E. P., Jr., and Williams, E. A note on basic sets of homogeneous harmonic polynomials. Proc. Amer. Math. Soc. **6** (1955), 769–770.

Les auteurs établissent des formules récurrentes pour les polynomes donnés dans un travail antérieur [mêmes Proc. **6** (1955), 191–194; MR **17**, 252] et précisent que dans le cas de deux variables ceux-ci se reduisent aux parties réelles et imaginaires de $(x_1+ix_2)^n$.

M. J. De Schwarz (Rome).

Komatu, Yûsaku. On boundary value problems for a rectangle. Kōdai Math. Sem. Rep. **7** (1955), 8–14.

This paper is concerned with the construction of a harmonic function from its boundary values on two opposite sides of a rectangle and from the values of its normal derivative on the remaining two sides. The author shows that with the help of elementary operations the problem may be converted into the usual Dirichlet problem for a rectangle with suitably modified boundary data. Explicit formulas are given. *Z. Nehari.*

Durand, Emile. Une expression simple de la solution du problème de Dirichlet dans le plan. C. R. Acad. Sci. Paris **241** (1955), 1452–1454.

Considering a simply-connected domain D and assuming that a conformal mapping g of a half-plane onto D is known the author represents the solution of the Dirichlet problem for D in terms of g. *A. Huber.*

Brelot, M. A new proof of the fundamental theorem of Kellogg-Evans on the set of irregular points in the Dirichlet problem. Rend. Circ. Mat. Palermo (2) **4** (1955), 112–122.

The author presents a self-contained, and basically elementary proof of Kellogg's Lemma: If Ω is a bounded domain, in three-space say, then the set E of irregular points for the Dirichlet problem has the property that each compact subset of E has zero capacity.

M. O. Reade (Ann Arbor, Mich.).

Albertoni, Sergio. Sulla risoluzione del problema di Neumann per l'equazione $\Delta_2 u-ku=f$. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18(87)** (1954), 400–432.

The author offers a detailed account of the Fredholm theory as it applies to the equation (1) $\nabla^2 u-\lambda^2 u=f$ in a bounded plane region of finite connectivity and boundary with continuous curvature, where λ is a given real or complex number and the boundary values B of the normal derivative are assigned. The function f satisfies a Hölder condition in the region, and the values B are assumed to be integrable and assumed continuously almost everywhere on the boundary. Let A be the values, assumed almost everywhere and integrable, of u on the boundary, and let v be the fundamental solution of the equation (1), which is expressible in terms of Bessel functions. Then inside and outside the region, respectively, Green's formulas give (2) $[A, B, f]=2\pi u$ and (3) $[A, B, f]=0$, where $[A, B, f]=\int\{A(\partial v/\partial n)-Bv\}ds-\int\int v\,dS$, n is the interior normal, ds the element of arc on the boundary and dS that of area. Amerio has proved [same Rend. (3) **9(78)** (1945), 79–102; MR **9**, 37] that if A, B, and f satisfy (3), then u as given by (2) satisfies (1) and the boundary conditions. Now (3), by suitable limit-taking yields formally a Fredholm equation on the boundary. The present paper is devoted to showing that this Fredholm equation is actually equivalent to (3) and that consequently the Fredholm theory yields the solution of (1). The method amounts to a rather detailed analysis of the kernel of the Fredholm equation. *J. W. Green.*

Satō, Tokui. Sur l'équation aux dérivées partielles $\Delta z=f(x, y, z, p, q)$. Compositio Math. **12** (1954), 157–177.

In this paper the author studies the differential equation given in the title, where $\Delta z(x, y)$ is the generalized Laplace operator:

$$\Delta z(x, y)\equiv\lim_{r\to 0}\frac{1}{2\pi r^2}\int_0^{2\pi}\{z(x+r\cos\theta, y+r\sin\theta)-z(x, y)\}d\theta,$$

and where $p\equiv\partial z/\partial x$, $q\equiv\partial z/\partial y$, under varying hypotheses on f and z. He obtains a maximum principle, existence and uniqueness theorems, and convergence theorems. The author considers "least" and "largest" solutions of the differential equation, reminiscent of certain analogues in potential theory [see T. Radó, Subharmonic functions, Springer, Berlin, 1937; in particular, ch. V]. The author also studies the dependence of the solution upon the parameter introduced as follows: $\Delta z=f(x, y, z, p, q, \lambda)$.

M. O. Reade (Ann Arbor, Mich.).

Duff, G. F. D. A tensor boundary value problem of mixed type. Canad. J. Math. **6** (1954), 427–440.

L'A. a montré antérieurement [Ann. of Math. (2) **57** (1952), 115–127; MR **13**, 986] qu'une variété de Riemann M, de classe C^∞ ainsi que sa frontière B, peut être prolongée en une variété compacte F de même classe et de même dimension N, par adjonction de son double $\tilde{M}$. Il a également indiqué, à la fin d'un mémoire antérieur [Canad. J. Math. **5** (1953), 524–535; MR **15**, 799] que les problèmes aux limites concernant l'équation de Laplace $\Delta\varphi=0$ dans M (où φ est une forme différentielle de degré p) peuvent être traités en considérant d'abord l'équation $\Delta\varphi+A\varphi=0$, où A est une matrice tensorielle de classe C^∞, définie positive dans $\tilde{M}$, nulle dans M, et définissant une application linéaire $\varphi\to A\varphi$ de l'espace des formes différentielles φ dans lui-même. Cette méthode est appliquée ici à la solution des problèmes aux limites (K) $t\varphi=t\xi$, $t\delta\varphi=t\delta\xi$, (M) $n\varphi=n\xi$, $nd\varphi=nd\xi$, (M) se ramenant à (K) pour une forme de degré $N-p$. Ces problèmes sont limites de ceux considérés par l'A. [ibid. **5** (1953), 196–210; MR **14**, 903]. On montre d'abord que ces problèmes ont une solution unique pour l'équation $\Delta\varphi+A\varphi=0$ quand A est définie positive dans M. La solution est construite au moyen de potentiels de simple et double couche, étendues à B, de la solution élémentaire g_A relative à F, et les densités de couche sont déterminées par une équation intégrale à laquelle on applique la méthode de Poincaré-Fredholm. Dans le cas où A s'annule sur M, on remplace g_A par la forme de Green de de Rham, selon la méthode exposée dans le 2ième mémoire cité; et on montre

ainsi que le problème (K) pour l'équation $\Delta\varphi=0$, admet une solution pourvu que $\int_B \partial\xi V^*\varphi=0$ pour tout champ harmonique φ satisfaisant à $t\varphi=0$ sur B. L'A. examine ensuite le cas particulier des formes fermées et cofermées; puis il établit l'existence d'un opérateur de Green P_K tel que $P_K\varphi$ est la forme harmonique résolvant le (K) problème pour les données frontières φ, et assujettie à $K\varphi=0$, où K est l'opérateur de projection sur l'espace des formes propres de (K). Il termine par quelques exemples de l'espace à 2 et 3 dimensions. *J. Lelong* (Zbl 55, 331).

Duff, G. F. D. On the potential theory of coclosed harmonic forms. Canad. J. Math. 7 (1955), 126–137.

Poursuivant ses recherches sur les problèmes aux limites associés à l'équation de Poisson pour les formes différentielles, l'A. cherche ici solutions φ satisfaisant à $\delta\varphi=\sigma$, où σ est une forme donnée. Utilisant ses résultats antérieurs [même J. 5 (1953), 196–210; MR 14, 903; voir aussi l'oeuvre analysée ci-dessus] ainsi que ceux de D. C. Spencer, il prouve, sous certaines conditions vérifiées par les données, l'existence d'une solution unique φ de $\delta d\varphi=\varrho$, $\delta\varphi=\sigma$, assujettie à diverses conditions aux limites, et orthogonale à tous les champs harmoniques solutions du problème homogène correspondant. Examen des cas où cette solution est dérivée ou co-dérivée. Comme cas particulier, l'A. établit l'existence d'une solution unique ϕ de $d\varphi=\varrho$, $\delta\varphi=\sigma$, satisfaisant à des conditions analogues. Enfin il étudie le problème mixte et le problème de Neumann relatifs au système $\delta d\varphi=\varrho$, $d\delta\varphi=\sigma$ (cas biharmonique). *J. Lelong* (Lille).

See also: Carathéodory, p. 446; Kudryavcev, p. 471; Heins, p. 473; Jörgens, p. 493; Davies, p. 496.

Series, Summability

Vermes, P. Infinite matrices summing every periodic sequence. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17 (1955), 627–633.

The problem of summability of divergent periodic sequences was introduced by R. H. C. Newton [same Proc. 57 (1954), 533–544, 545–549; MR 16, 691]. Newton restricted his investigations to regular matrix methods of summability, i.e. methods which sum convergent sequences to the value to which they converge. The author of the present paper does not restrict his work to regular matrices. Calling a matrix that sums every periodic sequence a P-Matrix the following theorem is proven. The matrix A is a P-Matrix if and only if at every point z of rational affix on the unit circle, and for $n=0, 1, 2, \cdots$ it is true that (1) the series $\sum_k a_{nk}z^k$ converges and $\sum_k a_{nk}z^k=a_n(z)$ tends to a finite limit. It is shown by examples that for P-Matrices the rows need not be absolutely convergent, the elements can be unbounded and the column limits need not exist. Additional restrictions are given for determining various subclasses of the P-Matrices, for example, those which are in addition regular. *V. F. Cowling* (Lexington, Ky.).

Siddiqi, Jamil A. On a theorem of Fejér. Math. Z. 61 (1954), 79–81.

Let $\Lambda=\{\lambda_{n,k}\}$, $n, k=0, 1, 2, \cdots$, $\lambda_{n,k}=0$ for $k>n$ and $\lambda_{n,0}=1$ be a triangular matrix. Let $\Delta\lambda_{n,k}=\lambda_{n,k}-\lambda_{n,k+1}$. A sequence (s_n) is said to be summable (Λ) if $\lim_{n\to\infty}\sum_{k=0}^n \Delta\lambda_{n,k}S_k$ exists. The conditions for the regularity of the method are (i) $\lambda_{n,k}\to 1$ as $n\to\infty$ for every fixed k and (ii) $\sum_{k=0}^n |\Delta\lambda_{n,k}|\leq M$, independently of n. Let $f(x)$ be of bounded variation in $(-\pi, \pi)$ and of period 2π. Let $\sum (b_n\cos nx-a_n\sin nx)=\sum c_n(x)$ be the conjugate Fourier series of $f(x)$. The author proves that the sequence $nc_n(x)$ is summable (Λ) provided the additional condition (iii) $\sum_{k=0}^n |\Delta^2\lambda_{n,k}|\to 0$ as $n\to\infty$, is satisfied. The (Λ)-limit is $\pi^{-1}[f(x+0)-f(x-0)]$. This generalises a theorem of Fejér that, under the same hypothesis, $nc_n(x)$ is summable (C, r) for $r>0$. The above general result includes the harmonic and Cesaro means as special cases. Note: In the proof, $\int_0^\pi |d\psi_n(t)|$ should be interpreted as $\int_{+0}^\pi |d\psi_n(t)|$. *V. Ganapathy Iyer* (Zbl 56, 62).

Grummich, Friedrich. Über die Gültigkeit des distributiven Gesetzes bei unendlichen Produkten. Math. Nachr. 13 (1955), 257–272.

Let $\{a_{nk}\}_{n\geq 1, k\geq 0}$ be a double sequence of complex numbers such that the series $\sum_{k=0}^\infty a_{nk}$ converges for each n, and the product $(\prod)$: $\prod_{n=1}^\infty(\sum_{k=0}^\infty a_{nk})$ converges to the number P. The author studies the "formal distributive development" of $(\prod)$. Let K be the set of all sequences $k=\{k(i)\,|\,1\leq i\leq\infty\}$ of non-negative integers. The product $(\prod)$ is said to be summable [conditionally summable] to the number S provided for each $k\in K$, the product $\prod_1^\infty a_{ik(i)}$ converges to a number $p(k)$, the set K^* of all $k\in K$ for which $p(k)\neq 0$ is countable, and there is an ordering $\{k_j\}$ of the k's in K^* such that the series $\sum_{j=1}^\infty p(k_j)$ is absolutely convergent [conditionally convergent] to S. Some of the results are as follows: (1) If $(\prod)$ is summable to S, then either $S=P$ or $S=0$. (2) If $\prod_{n=1}^\infty a_{n0}$ converges to $a\neq 0$, then $(\prod)$ is summable (and necessarily to P) if and only if the double series $\sum_{n,k=1}^\infty (a_{nk}/a_{n0})$ is absolutely convergent. (3) Suppose $p(k)$ exists for each $k\in K$, that the product $(\prod)$ is not unconditionally summable, and that $|a_{n0}|\geq|a_{nk}|$ for all n, $k\geq 1$. Then $(\prod)$ is conditionally summable to P if and only if for each $\varepsilon>0$ there exists $n_0(\varepsilon)$ such that $|a_{nk}|<\varepsilon$ for all (n, k) with $n>n_0(\varepsilon)$ and $k\geq 1$. *V. L. Klee.*

McArthur, C. W. Some applications of theorems on uniform Cauchy points to infinite series. Proc. Amer. Math. Soc. 6 (1955), 603–612.

The notions of uniform Cauchy points and points of uniform convergence [which have been studied earlier by W. L. Gordon and the author Amer. J. Math. 74 (1952), 764–768; MR 14, 362] are here applied to certain problems concerning rearrangements of infinite series in pseudometric semigroups. The principal theorem has several interesting consequences (too complicated to state here) which lead by successive specialization to the following sharpened form of a theorem of Agnew [Bull. Amer. Math. Soc. 46 (1940), 797–799; MR 2, 89] and Sengupta [Proc. Amer. Math. Soc. 1 (1950), 71–75; MR 11, 346]: Let s be a sequence of real numbers and $P(N)$ the space of all permutations of natural numbers under the usual Fréchet metric. Let $E_{\pm\infty}(s)$ be the set of all $t\in P(N)$ such that $\limsup_n \sum_{i=1}^n s(t(i))=\pm\infty$. Then if $E_{+\infty}(s)$ or $E_{-\infty}(s)$ is nonempty, it is a residual dense G_δ set in $P(N)$, and if both sets are nonempty their intersection is a residual dense G_δ set. *V. L. Klee* (Los Angeles, Calif.).

Orts, José M.ª. On some nonlinear recurrence relations. Rev. Mat. Hisp.-Amer. (4) 15 (1955), 115–120. (Spanish)

The author is concerned with Taylor series $\sum a_n z^n$ whose coefficients satisfy a non-linear recurrence relation.

No general theorems are proven. However, several examples of this sort are presented. Consider, for example, the Riccati equation, $y'=ky^2+p(x)$, where $p(x)$ is a polynomial and k is a constant. If one assumes $y=\sum a_n z^n$ is a solution then

$$a_{n+1}=\frac{k}{n+1}(a_0a_n+a_1a_{n-1}+\cdots+a_na_0)$$

is valid for all integers n larger than the degree of the polynomial $p(x)$. *V. F. Cowling* (Lexington, Ky.).

★ **Markuschewitsch, A. I. Rekursive Folgen.** Deutscher Verlag der Wissenschaften, Berlin, 1955. 48 pp. DM 2.80.

Translation of Markuševič's popular expository work, Vozvratnye posledovatel'nosti [Gostehizdat, Moscow, 1951].

Życzkowski, M. Theorems on the errors of approximating a function by the first terms of its power series. Prace Mat. 1 (1955), 371–392. (Polish. Russian and English summaries)

Let $f(x)=\sum_0^\infty a_nx^n$ be convergent on the interval $-r\leq x\leq r$ and denote

(1) $|R_p(x)|=|\sum_p^\infty a_nx^n|$, (2) $|P_p(x)|=|R_p(x)|/|f(x)|$.

Conditions are given for (1) and (2) to be nondecreasing on an interval $0\leq x\leq x_m$, where $x_m\leq r$. A few cases are considered. *S. Kulik* (Columbia, S.C.).

Gonçalves, J. Vicente. Les développements de tg x et cot x. Univ. Lisboa. Revista Fac. Ci. A. (2) 3 (1954–1955), 331–332.

Macon, N. A continued fraction for e^z. Math. Tables Aids Comput. 9 (1955), 194–195.

Oberhettinger, F. On asymptotic series for functions occurring in the theory of diffraction of waves by wedges. J. Math. Phys. 34 (1956), 245–255.

The author investigates the asymptotic expansions, for large values of $k\varrho$, of the function

$$\frac{1}{2\alpha}\sin\frac{\pi\delta}{\alpha}\int_0^\infty\frac{H(x)dx}{\cosh\pi x/\alpha-\cos\pi\delta/\alpha},$$

where $H(x)$ has one or other of the following forms:

$\exp(-ik\varrho\cosh x)$, $H_0^{(2)}[k(\varrho^2+\varrho'^2+2\varrho\varrho'\cosh x)^{\frac{1}{2}}]$, $(1/R)\exp(-ikR)$,

where $R^2=\varrho^2+\varrho'^2+2\varrho\varrho'\cosh x+(z-z')^2$. The behaviour of the function when $k\varrho$ is small is also discussed.

E. T. Copson (St. Andrews).

★ **Berghuis, Johan. The method of critical regions for twodimensional integrals and its application to a problem of antenna theory.** Thesis, Technische Hogeschool te Delft, 1955. 79 pp.

This paper studies the integral

$$\int_D\int g(w,x,y)\exp[if(w,x,y)]dxdy$$

for the case that the parameter w is large and positive. An application is made to the following problem in antenna theory. Find the potential of a vertical dipole placed in the boundary layer of two infinite media of different permeability, each filling a half-space. The mathematical methods lean heavily on the work of van der Corput.

A. E. Heins (Pittsburgh, Pa.).

See also: Mikolás, p. 481.

Interpolation, Approximation, Orthogonal Functions

★ **Bruwier, L. Sur un problème d'interpolation.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 24–28. Fédération belge des Sociétés Scientifiques, Bruxelles.

Pour $x_1<x_2<\cdots<x_{n+1}$, $\psi(x)>0$, et $f(x)$ continue, recherche d'une fonction $\varphi(x)$ continue telle que:

$$\int_{x_j}^{x_{j+1}}\varphi\psi dx=\int_a^{x_{j+1}}f\psi dx\ (j=1,\cdots,n),$$

et satisfaisant à diverses autres conditions, comme, par exemple, d'être un polynome de degré $(n-1)$. Examen de cas particuliers. *J. Favard* (Paris).

Müller, Max. Über Interpolation mittels ganzer rationaler Funktionen. Math. Z. 62 (1955), 292–309.

Der. Verf. betrachtet die allgemeinere Interpolationsaufgabe, bei der an den Interpolationsstellen $x_0, x_1, \cdots, x_s$ das Interpolationspolynom $P_n(x)$ mit der interpolierten Funktion $f(x)$ ausser in den Funktionswerten auch in den Werten der Ableitungen bis zu einer gewissen (von Interpolationsstelle zu Interpolationsstelle auch verschieden wählbaren) Ordnung hin übereinstimmen soll. Das Interpolationspolynom wird bestimmt durch

$$P_n(x)=\sum_{\sigma=0}^{s}\sum_{\alpha=0}^{\alpha_\sigma-1}f^{(\alpha)}(x_\sigma)L_{\sigma\alpha}(x),$$

wo $\alpha_0, \alpha_1, \cdots, \alpha_s$ beliebige natürliche Zahlen sind mit $\sum_{\sigma=0}^{s}\alpha_\sigma=n+1$. $L_{\sigma\alpha}(x)$ ist ein bestimmtes Polynom höchstens n-ten Grades das an der Stelle x_τ $(\tau\neq\sigma)$ mit samt den $\alpha_\tau-1$ ersten Ableitungen verschwindet; die Stelle x_τ ist somit mindestens α_τ-fache Nullstelle von $L_{\sigma\alpha}(x)$. Das Restglied $R_n(x)=f(x)-P_n(x)$ wird in drei verschiedenen Formen gegeben; erstens als Übertragung der Cauchyschen Fehlergliedformel für das Lagrangesche Interpolationspolynom auf den allgemeineren Fall, zweitens in Verallgemeinerung der Wegnerschen Form $\int_0^1 f^{(\tau)}(t)K_\tau(t,x)dt$ (τ ist eine beliebige ausgewählte der Zahlen $1, 2, \cdots, n+1$) (der Verf. leitet eine Darstellung des Kernes $K_\tau(t,x)$ mittels ganz einfachen, explizit angebbaren Funktionen, als ein lineares Aggregat der Polynome $L_{\sigma\alpha}(x)$ ab), und drittens durch eine weitere Umformung des Kernes $K_\tau(t,x)$ mittels Bernoullischen Polynomen. Schliesslich wird als Anwendung die allgemeine Quadraturformel von A. Kneschke [J. Reine Angew. Math. 187 (1949), 115–128, S. 120; MR 12, 13] abgeleitet.

S. C. van Veen (Delft).

Butzer, P. L. Summability of generalized Bernstein polynomials. I. Duke Math. J. 22 (1955), 617–623.

Let

$$P_n(x;f)=\sum_{k=0}^{n}(n+1)\binom{n}{k}x^k(1-x)^{n-k}\int_{k/(n+1)}^{(k+1)/(n+1)}f(t)dt\ (0<x<1),$$

$$Q_n(x;f)=(n+1)P_n(x;f)-nP_{n-1}(x;f).$$

The convergence and summability properties of $\{Q_n\}$ are investigated. In particular, it is found that the sequence

$\{Q_n(x_0; f)\}$ may diverge for a continuous function $f(t)$. For continuous $f(t)$, $Q_n(x; f)=O(\sqrt{n})$; this bound can not be improved. If $f(x)$ is bounded and $f''(x_0)$ exists, then $Q_n(x_0, f)\to f(x_0)$. *W. H. J. Fuchs* (Ithaca, N.Y.).

Schweizer, Berthold. A symmetric generalization of the Lagrange interpolation formula. J. Math. Phys. **34** (1955), 157–159.

Soit $f(z)$ une fonction dont on connaît les valeurs en n points différents a_i $(i=1, \cdots, n)$, ainsi que celles de ses dérivées jusqu'à l'ordre k; la notation: $[a_1^p, \cdots, a_{i-1}^p, a_i^{p+1}, a_{i+1}^p, \cdots, a_n^p]=\Delta_i^p$ désigne la différence divisée de f, où les arguments a_i sont répétées p fois, sauf a_i qui est répété $(p+1)$ fois; on pose: $P(z)=(z-a_1)\cdots(z-a_n)$, alors le polynome prenant les mêmes valeurs que f aux points a_i, et y admettant les mêmes dérivées jusqu'à l'ordre k est:

$$\sum_{p=0}^{k}\left\{\sum_{i=1}^{n}\frac{\Delta_i^p}{z-a_i}\frac{P(z)}{P'(a_i)}\right\}[P(z)]^p.$$

J. Favard (Paris).

Hirschman, I. I., Jr. Weighted quadratic norms and Legendre polynomials. Canad. J. Math. **7** (1955), 462–482.

Let w_n denote the normalized Legendre polynomial of degree n. The (fixed) complex sequence $\{t_n\}_0^\infty$ defines a linear transformation T of (part of) $L^1(-1, 1)$ into $L^1(-1, 1)$ if one sets $g=Tf$ whenever f, $g\in L^1(-1, 1)$ and $\int_{-1}^{1} gw_n dx=t_n\int_{-1}^{1} fw_n dx$ $(n=0, 1, 2, \cdots)$. Let

$$\mathfrak{N}_{\beta,\alpha}(f)=\left[\int_{-1}^{1}(1+x)^\beta(1-x)^\alpha|f(x)|^2dx\right]^{1/2}.$$

The author's main result is the following: If $|t_n|\leqq C$ and

$$\sum_{k=2^n}^{2^{n+1}}|t_k-t_{k-1}|\leqq C \quad (n=0, 1, 2, \cdots),$$

then $\mathfrak{N}_{\beta,\alpha}(Tf)\leqq CD(\beta, \alpha)\mathfrak{N}_{\beta,\alpha}(f)$ for $-\frac{1}{2}<\alpha$, $\beta<\frac{1}{2}$, where $D(\beta, \alpha)$ is independent of f; the restriction on α and β is essential.

As might be expected, the proof is not particularly easy. A third of the paper is devoted to the derivation of the estimate

$$E'(\alpha)\leqq\mathfrak{N}_{0,\alpha}{}^{-2}(f)\sum_{k>j}|b_{kj}-b_{jk}|^2|k-j|^{-1-2\alpha}(k+j)^{-1}\leqq E''(\alpha), \qquad 0<\alpha<\tfrac{1}{2},$$

where $b_{kj}=(j+\frac{1}{2})^{1/2}\int_{-1}^{1} fw_k dx$, and another third is needed to establish the main result for $0<\alpha<\frac{1}{2}$ and $\beta=0$. There follow three pages extending the result to $-\frac{1}{2}<\alpha$, $\beta<\frac{1}{2}$.

The author takes $t_0=0$ and $t_n=[n(n+1)]^{-\sigma/2}$ for $n>0$ to get the linear transformation T_σ which is a fractional integration (differentiation) operator if $\sigma>0$ (<0). He closes the paper by applying his main result above and some earlier results on multiplier transformations [J. Analyse Math. **2** (1953), 209–218; Mem. Amer. Math. Soc. no. **15** (1955); MR **15**, 295, 1139; **17**, 257] to obtain the following analog of a theorem of Hardy and Littlewood [A. Zygmund, Trigonometrical series, Warsaw-Lvov, 1935, p. 227]: If $\int_{-1}^{1} f dx=0$, $\sigma>0$, $\alpha<\frac{1}{2}$, $\beta<\frac{1}{2}$, $\alpha'=\alpha-\sigma$, $\beta'=\beta-\sigma$, $\alpha'>-\frac{1}{2}$, and $\beta'>-\frac{1}{2}$, then

$$\mathfrak{N}_{\alpha',\beta'}(T_\sigma f)\leqq B(\alpha,\beta,\sigma)\mathfrak{N}_{\alpha,\beta}(f).$$

A. E. Livingston (Seattle, Wash.).

Leja, F., et Opial, Z. Un lemme sur les polynômes de Lagrange. Ann. Polon. Math. **2** (1955), 73–76.

On donne une suite triangulaire de nombres complexes $\zeta_j^{(n)}$, $0\leqq j\leqq n$, $\zeta_j^{(n)}\neq\zeta_k^{(n)}$ pour $j\neq k$, admettant z_0 pour point d'accumulation. Soit $L_n^{(j)}(z)$ le polynôme de degré n égal sur $\zeta_k^{(n)}$ à 0 ou 1 suivant que $j\neq k$ ou $j=k$. Soit $M_n(z_0, r)=\max\{0, |L_n^{(j)}(z_0)|\}$ pour $|\zeta_j^{(n)}-z_0|<r$. On a, quel que soit $r>0$, $\limsup (M_n(z_0, r)^{1/n}\geqq 1$ $(n\to\infty)$. D'ailleurs, on peut choisir $\{\zeta_j^{(n)}\}$, z_0 et r de façon que cette lim sup prenne une valeur arbitraire $\geqq 1$. *J. P. Kahane.*

Fekete, Michael. Approximation by polynomials with Diophantine side-conditions. Riveon Lematematika **9** (1955), 1–12. (Hebrew. English summary)

Let G be the class of polynomials whose coefficients are Gaussian integers, and denote by C a bounded closed set in the complex z-plane. This paper is concerned with the conditions under which a continuous function $f(z)$ on C can be uniformly approximated by polynomials of G. The author proves the following three theorems: (a) If there exists an $f(z)$ (not belonging to G) which can be approximated in this way, then the transfinite diameter (outer conformal radius) of C is $\leqq 1$. (b) Only a finite number of points of C can be roots of equations $P(z)=0$, $P(z)\in G$. If the set S of these points is not empty, then the Lagrange interpolation polynomial $P(z; S)$ of a function $f(z)$ of the above type belongs to G. (c) The necessary conditions given in (a) and (b) are also sufficient for the uniform approximability of $f(z)$ by polynomials of G.

Z. Nehari (Pittsburgh, Pa.).

Remez, E. Ya. On graphic-analytic solution of some problems of Čebyšev approximation. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 409–412. (Russian)

Remez, E. Ya. The method of graphic-analytic solution of certain problems of Čebyšev approximation. Ukrain. Mat. Ž. **7** (1955), 71–90 (1 plate). (Russian)

Application of the characteristics of that curve f out of a given family F which has smallest maximum deviation from a given set S [Motzkin, Bull. Amer. Math. Soc. **55** (1949), 789–793; MR **11**, 101] to the cases where F consists of all straight lines, or of all straight lines through the origin. For finite S inspection of its graph gives the points of maximum deviation, and an elementary follow-up computation furnishes the coefficients of the equation of f. *T. S. Motzkin* (Los Angeles, Calif.).

Paszkowski, S. Sur l'approximation uniforme avec des noeuds. Ann. Polon. Math. **2** (1955), 118–135.

Let the functions $p_1(x)$, $\cdots$, $p_n(x)$ be continuous in a closed interval I' and such that every non-trivial linear combination $w(x)=a_1p_1(x)+\cdots+a_np_n(x)$ has at most $n-1$ distinct roots in I'. Let I be a subinterval of I', t_1, $\cdots$, t_m points in I'. The author considers the problem of best uniform approximation in I of functions $f(x)$ defined in I' and continuous in I by functions $w(x)$ subject to the conditions $w(t_j)=f(t_j)$, $j=1, \cdots, m$. Generalizations of theorems of Tchebycheff and de la Vallée Poussin are obtained. For example, $w(x)$ of best approximation has the following property. There are l points in I where $|f(x)-w(x)|$ attains its maximum, which together with the t_j form a sequence $s_1<\cdots<s_{l+m}$. Define inductively $\eta_i=\text{sign}\,[f(s_i)-w(s_i)]$ if $f(s_i)-w(s_i)\neq 0$, $\eta_i=-\eta_{i-1}$ in the opposite case. Then there are at least n changes of sign in the sequence s_i. *G. G. Lorentz* (Detroit, Mich.).

van Kampen, N. G. Completeness of stationary scattering states. II. Physica **21** (1955), 579–588.

In part I [Physica **21** (1955), 127–136; MR **16**, 1018] the completeness in $L^2(0, \infty)$ of $\sin(kr+\eta(k))$ is con-

sidered. Here the author states that a sufficient condition for completeness is that $S(x)=e^{2i\varphi(x)}$ can be written as $p_+(x)/p_-(x)$, where $p_+(x)$ is the boundary value of a function $p_+(z)$ holomorphic in the upper half complex z-plane, and $p_-(x)$, the lower half plane. A more restricted case in discussed in detail. *N. Levinson.*

Indritz, J. Applications of the Rayleigh Ritz method to variational problems. Pacific J. Math. **5** (1955), 765–797.

Let R be a plane bounded domain, whose boundary consists of a finite number of non-intersecting simply closed regular arcs of class C^k. The norm

$$(1) \qquad \|v\|^2=\iint_R (av_x^2+bv_y^2+cv^2)dxdy$$

is introduced in the linear manifold V of the functions v of class C^k in $R+\Gamma$ and vanishing on Γ; a and b are positive integrable functions on $R+\Gamma$; c a non-negative integrable function on $R+\Gamma$. The closure of V is a Hilbert space H. Let $G(x, y)$ be a function of class C^k on $R+\Gamma$, vanishing on Γ, positive in R, with $\partial G/\partial\nu\geqq\delta>0$ on Γ. It is proved that the set of functions $G\tau$, where τ is a polynomial in x and y, is dense in H. Let $\{f_i\}$ be the set obtained by orthonormalising the set $\{Gx^hy^k\}$ with respect to the norm (1).

For any $v\in V$ it is proved that

$$\|v-\sum_{i=1}^{n} b_if_i\|^2\leqq\frac{\Theta(n)}{n^{k-2}};$$

b_i is the Fourier coefficient (v, f_i) and $\Theta(n)$ a function depending on v such that $\lim_{n\to\infty}\Theta(n)=0$. If $k\geqq0$, then

$$(2) \qquad \left|v-\sum_{i=1}^{n} b_if_i\right|=O\left(\left(\frac{\Theta(n)}{n^{k-2}}\log\frac{n}{(\log n)^N}\right)^{\frac{1}{2}}\right);$$

N is any fixed positive constant. Pointwise bounds for the first derivatives, analogous to (2), are indicated for k large enough.

Some extensions of these results are obtained in connection with the more general norm

$$\iint_R (av_x^2+bv_y^2+cv^2)dxdy \quad (p\geqq1).$$

G. Fichera (Rome).

van der Corput, J. G. Asymptotics. II. Elementary methods. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 139–150.

[For part I see same Proc. **57** (1954), 206–217; MR **16**, 588.] In section 1 a particular kind of enveloping series is investigated. A number s is called enveloped by the formal series $\sum a_n$ $(n\geqq0)$ with respect to the circular segment E_ζ bounded by the line segment $(0, 1)$ and by the circular arc with end-points 0 and 1 which makes at the origin an angle $-\zeta$ with the positive real axis, if it is possible to find for each integer $n\geqq0$ an element θ_n of E_ζ such that $s=\sum_0^{n-1} a_k+\theta_na_n$. Notation: $s\propto\sum_{n=0}^{\infty} a_n(E_\zeta)$. In section 2 these results are applied to the asymptotic expansion of $\log\Gamma(z)$. Theorem 2.2 asserts: For $|\arg z(=\zeta)|<\pi$, and ϱ an arbitrarily chosen number with $|\varrho|<\frac{1}{2}\pi$, $|\varrho-\zeta|<\frac{1}{2}\pi$, the remainder term r_n in this expansion is $\leqq\sigma(\cos\varrho)^{-2n-2}$ times the first neglected term, where $\sigma=1$ for $|\varrho-\zeta|<\frac{1}{4}\pi$, $\sigma=1/|\sin 2(\varrho-\zeta)|$ for $\frac{1}{4}\pi\leqq|\varrho-\zeta|<\frac{1}{2}\pi$. [Remark by the reviewer: It may be noted that $\sigma=1/|\sin 2(\varrho-\zeta)|$ for $|\varrho-\zeta|<\frac{1}{2}\pi$ was already obtained by Lindelöf, Le calcul des résidus ⋯, Gauthier-Villar, Paris, 1905, p. 100.]

In section 3, which deals with the Maclaurin expansion, $f^{(n)}(z)$ $(n\geqq1)$ is assumed to belong for each point z on the line segment $(0, w)$ (w real or complex $\neq0$) to a convex set E_n. Then for each point z on the line segment $(0, w)$

$$f(z)=\sum_{h=0}^{n-1}\frac{f^{(h)}(0)}{h!}z^h+\frac{\lambda_nZ^n}{n!},$$

where λ_n belongs to E_n. In section 4 asymptotic expansions are obtained by integration by parts. By application of these results the asymptotic expansion of integrals, containing among others absolutely monotonic or completely monotonic functions, is obtained in section 5. (A function $f(x)$ is called absolutely monotonic in the interval j, if $f^{(n)}(x)\geqq0$ $(n=0, 1, 2, \cdots)$ and it is called completely monotonic in j, if $(-1)^nf^{(n)}(x)\geqq0$ $(n=0,1\ 2, \cdots)$. In the last section 6 some well-known theorems are used, giving sufficient conditions for the validity of:

$$\lim_{h\to\infty}\sum_{0\leqq n<m_h} f_n(h)=\sum_{n=0}^{\infty}\lim_{h\to\infty} f_n(h) \quad (m_h=\infty \text{ or } \to\infty \text{ for } h\to\infty).$$

This section contains many examples from the theory of ultraspherical polynomials, polynomials of Fejér etc.: e.g.

$$h^{\lambda-1}P_h(\cos\theta)=\frac{2^{\lambda-1}}{\Gamma(\lambda)}(\sin\theta)^{-\lambda}\cos\left\{(h+\lambda)\theta-\frac{\lambda\pi}{2}\right\}+\varepsilon_h,$$

where $\varepsilon_h\to0$ as $h\to\infty$. *S. C. van Veen* (Delft).

Andreoli, Giulio. Sistemi ortogonali di Walsh; matrici di Hadamard; trasformazioni di sistemi ortogonali. Ricerca, Napoli **6** (1955), no. 1, 3–17.

En généralisant une méthode de composition des matrices carrées, due à Kronecker, et en l'appliquant aux matrices de Hadamard (celles dont les éléments sont en module $\leqq1$, et dont le déterminant de la valeur absolue a la valeur maximum $n^{n/2}$), l'auteur obtient des systèmes orthogonaux généralisant celui de Walsh. *J. Favard.*

See also: Šaginyan, p. 471.

Trigonometric Series and Integrals

Izumi, Shin-ichi. Some trigonometrical series. XV. Proc. Japan Acad. **31** (1955), 399–401.

Let $f(x)$ belong to class Lip (α, p), $p\geqq1$, $0<\alpha<1$, and either $\alpha p>1$ or $\alpha p<1$. If $s_n(x)$ denotes the nth partial sum of the Fourier series of $f(x)$, then $f(x)-s_n(x)=O(n^{(1-\alpha p)/p})$ uniformly almost everywhere.

P. Civin (Eugene, Ore.).

Satô, Masako. Lacunary Fourier series. I. Proc. Japan Acad. **31** (1955), 402–405.

Soit $f(x)\sim\sum(a_{n_k}\cos n_kx+b_{n_k}\sin n_kx)$. L'auteur donne des conditions portant sur la suite $\{n_k\}$ et sur le comportement de f au voisinage d'un point, garantissant $|a_{n_k}|+|b_{n_k}|=O(n_k^{-\alpha})$ $(0<\alpha<1$ donné). Exemple de telles conditions (th. 2): soit $0<\beta<(1-\alpha)/2$, $\gamma>2/(1-\beta)$: on suppose $n_k=[k^\gamma]$; soit $\omega(\tau, h)=\tau^{-1}\int_0^\tau|f(t)-f(t\pm h)|dt$: on suppose $\omega(\tau, h)$ bornée pour $\tau\geqq h^\beta$ et $\omega(h^\beta, h)=O(h^\alpha)$ (rectifier ainsi (3) p. 404). *J. P. Kahane* (Montpellier).

Karamata, J., et Tomić, M. Sur la sommation des séries de Fourier. Glas Srpske Akad. Nauka **206**. Od. Prirod.-Mat. Nauka (N.S.) **5** (1953), 89–126. (Serbo-Croatian. French summary)

The authors consider the summation of Fourier series

by means of the form $\frac{1}{2}\lambda_0^n a_0+\sum_{\nu=1}^{\infty}\lambda_\nu^n(a_\nu\cos\nu x+b_\nu\sin\nu x)$. When $\{\lambda_\nu^n\}$ is a triangular matrix, Nikolsky [Izv. Akad. Nauk SSSR. Ser. Mat. **12** (1948), 259–278; MR **10**, 247] gave necessary and sufficient conditions for the method to be Fourier effective (which means roughly that it sums all Fourier series at Lebesgue points), as well as some special sufficient conditions. Further sufficient conditions, including both those of Nikolsky and of Hille and Tamarkin [Trans. Amer. Math. Soc. **34** (1932), 757–783] were given by B. Sz.-Nagy [Acta Sci. Math. Szeged **12**, Pars B (1950), 204–210; MR **11**, 656]. The authors extend these results to general matrices and also give more general special sufficient conditions. [Cf. the following review.] *R. P. Boas. Jr.* (Evanston, Ill.).

Karamata, J., et Tomić, M. Sur la sommation des séries de Fourier des fonctions continues. Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 123–138.

The authors correct the results of their paper reviewed above. Their main theorem now reads as follows. Consider the convergence-factor summation method under which the sum of $\sum u_\nu$ is $\lim_{n\to\infty}\sum_{\nu=0}^{\infty}\lambda_{\nu,n}u_\nu$. A set of necessary and sufficient conditions for the method to sum the Fourier series of all continuous functions is as follows: $\lambda_{\nu,n}\to 1$ for each ν; if $K_{m,n}(t)=\frac{1}{2}\lambda_{0,n}+\sum_{\nu=1}^{m}\lambda_{\nu,n}\cos\nu t$, then $\int_0^\pi|K_{m,n}(t)|dt\leq M_n$ (independent of m); if $\bar{K}_n(x)=\lim_{m\to\infty}\int_0^x K_{m,n}(t)dt$, then $\int_0^\pi|d\bar{K}_n(t)|=O(1)$. A more usable set of sufficient conditions is $\lambda_{\nu,n}\to 1$, $\lambda_{\nu,n}=O(1/\log\nu)$, and

$$\sum_{\nu=0}^{\infty}{}'\frac{(m+\nu)|m-\nu|}{m}\log\frac{m+\nu}{|m-\nu|}|\Delta^2\lambda_{\nu,n}|\leq C$$

for some m (either constant or tending to ∞ with n), with C independent of n and $\Delta^2\lambda_{\nu,n}=\lambda_{\nu,n}-2\lambda_{\nu+1,n}+\lambda_{\nu+2,n}$; here $\sum'$ omits the term $\nu=m$. *R. P. Boas, Jr.*

Albrycht, J. A generalization of a Zygmund-Bernstein theorem. Ann. Polon. Math. **2** (1955), 64–66.

Let N be one of a pair of functions complementary in the sense of Birnbaum and Orlicz [Studia Math. **3** (1931), 1–67]. Let E be the set of measurable complex-valued functions $y(t)$ such that

$$\limsup_{T\to\infty}(2T)^{-1}\int_{-T}^{T}N(|y(t)|)dt\leq 1,$$

and let

$$\|x\|=\sup_{y\in E}\left\{\limsup_{T\to\infty}(2T)^{-1}\int_{-T}^{T}x(t)y(t)dt\right\}.$$

Then if $\lambda_{-n}<\lambda_{-n+1}<\cdots<\lambda_n$, $|\lambda_{-n}|=|\lambda_n|$ and $f(t)=\sum_{-n}^{n}a_k e^{i\lambda_k t}$, one has the inequality $\|f'\|\leq\lambda_n\|f\|$. *P. Civin* (Eugene, Ore.).

Integral Transforms, Operational Calculus

★**Hirschman, I. I., and Widder, D. V. The convolution transform.** Princeton University Press, Princeton, N. J., 1955. x+268 pp. $5.50.

Ce livre fait le point sur des sujets particulièrement étudiés ces dernières années par les auteurs [voir entre autres MR **9**, 89–90; **10**, 36, 371; **12**, 175, 497, 605, 823; **13**, 30–31, 457; et surtout **11**, 350] et prend la suite du livre de Widder, The Laplace transform [Princeton, 1941; MR **3**, 232]. Il traite des transformations $\varphi\to f$ définies par

$$f(x)=\int_{-\infty}^{\infty}G(x-t)\varphi(t)dt, \tag{1}$$

$$G(t)=(2\pi i)^{-1}\int_{-i\infty}^{i\infty}(E(s))^{-1}e^{st}dt, \tag{2}$$

$$E(s)=\exp(-cs^2+bs)\prod_k[(1-s/a_k)\exp(s/a_k)] \tag{3}$$

les a_k et b réels, $c\geq 0$, et $\sum a_k^{-2}<\infty$. Moyennant un changement de variables, (1) contient comme cas particulier les transformations de Laplace, Stieltjes, Stieltjes généralisée, Laplace itérée, Stieltjes itérée, Meijer (toutes pour $c=0$) et Weierstrass (pour $a_k=b=0$).

Formellement (chap. I), l'inversion de (1) se fait par

$$E(D)f(x)=\varphi(x) \tag{4}$$

et G est une fonction de Green (solution élémentaire) pour (4).

Si E est un polynôme (chap. II), G est un „noyau fini": c'est une fonction de fréquence dont la fonction caractéristique est E^{-1}, la valeur moyenne $-\sum a_k^{-1}$ et la variance $\sum a_k^{-2}$.

Les fonctions (3) sont les limites uniformes de polynômes P_n à racines réelles ($P_n(0)=1$) sur un segment de l'axe imaginaire (légère modification d'un théorème de Laguerre-Polya). D'après le „théorème de continuité" de Paul Lévy, les fonctions (2) sont donc les dérivées des fonctions de distribution limites de primitives de noyaux finis. Un sens précis est donné à (4) quand $c=0$, ainsi que l'interprétation de (4) dans les cas particuliers susmentionnés (chap. III).

Le chap. IV est consacré aux idée de Schoenberg [Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 11–17; MR **8**, 319]. Pour qu'une fonction de fréquence φ soit „variation diminishing" (c.à.d. que la convolution avec φ d'une fonction continue et bornée ne puisse pas en augmenter le nombre des changement de signe), il faut et il suffit qu'elle soit presque partout de la forme (2). On en déduit que $-\log G(t)$ est convexe, et que $G^{(n)}(t)$, si elle existe, admet n changements de signe. Les fonctions (2) sont également caractérisées comme fonctions de fréquence totalement positives.

Le comportement à l'infini de $G(t)$ et de ses dérivées successives (chap. V) dépend des nombres $\alpha_1=\max[a_k,-\infty]$ ($a_k<0$) et $\alpha_2=\min[a_k,+\infty]$ ($a_k>0$). Une étude spéciale est faite des cas $\alpha_1=-\infty$ (ou $\alpha_2=\infty$); l'allure pour $t\to\infty$ (ou $t\to-\infty$) diffère alors suivant qu'on a $c>0$ ou $=0$, $\sum|a_k|^{-1}<\infty$ ou $=\infty$.

Ainsi apparaissent, quand $c=0$, trois classes de fonctions $G(t)$. Pour chacune de ces classes, les auteurs précisent l'énoncé du théorème d'inversion (4), en affaiblissant autant que possible les hypothèses portant sur φ. Ils énoncent également les théorèmes d'inversion quand $\varphi(t)dt$ est remplacé dans (1) par $d\alpha(t)$. Tous les cas connus d'application de (4) (avec $c=0$) sont, d'après les auteurs, contenus dans ce chapitre (chap. VI).

La „théorie de la représentation" (chap. VII) consiste en conditions nécessaires et suffisantes pour que f soit de la forme (1), φ satisfaisant certaines conditions. Exemple: $b=c=0$; pour que f soit de la forme (1) avec $\varphi\in L_p$ ($1<p\leq\infty$), il faut et il suffit que $\|\prod_1^n(1-D/a_k)f(x)\|_p<M$ indépendant de n.

Le chap. VIII est consacré à la transformation de Weierstrass: $E(s)=\exp(-s^2)$ et $G(x)=(4\pi)^{-\frac{1}{2}}\exp(-x^2/4)$. Les principaux problèmes sont encore ceux de l'inversion

et de la représentation. On définit

$$\exp(-tD^2)f(x)=(2\pi i)^{-1}\int_{\delta-i\infty}^{\delta+i\infty} K(s-x,t)f(s)ds$$

avec $K(s,t)=(\pi/t)^{\frac{1}{2}}\exp(s^2/4t)$ et $\exp(-D^2)f(x)=\lim \exp(-tD^2)f(x)$ quand $t\uparrow 1$. Alors le théorème d'inversion garde la forme (4), tandis que les théorèmes de représentation font intervenir $\exp(-tD^2)f(x)$ $(0<t<1)$ au lieu des produits $\prod_1^n$ du chap. VII.

Lorsque, au lieu de (3), on a

$$E(s)=\prod(1-s^2/a_k^2),\ a_k>0,\ \lim k/a_k=\Omega,$$

l'inversion de (1) se fait formellement par

$$\varphi(t)=(2\pi i)^{-1}\int_C K(z)f(t+z)dz, \tag{5}$$

où $K(z)$ est la transformée de Laplace-Borel de $E(s)$, et C une courbe entourant le segment $[-i\pi\Omega, i\pi\Omega]$. (5) n'a pas toujours un sens, mais on a des formules de ce type (chap. IX) faisant intervenir $\int_{C(\varrho)} K(z)f(t+\varrho z)dz$, avec $0\leqq\varrho<1$, et $C(\varrho)$ dans la bande $|\Im z|<\pi\Omega/\varrho$. Les auteurs donnent également des théorèmes de représentation portant sur ces integrales.

Le chap. X est consacré à des „miscellaneous topics": polynômes de Bernstein, comportement à l'infini des transformées de convolution, analyticité des noyaux, classes quasi-analytiques. *J. P. Kahane.*

Calderón, A. P., and Devinatz, A. On Fourier-Stieltjes transforms. Canad. J. Math. 7 (1955), 453–461.

The Fourier-Stieltjes transformation $\varphi\to\Phi$ where $\Phi(x)=\int e^{ixt}d\varphi(t)$ is a 1-1 mapping of the class $\mathfrak{M}$ of bounded nondecreasing functions on the real lines (normalised so that $\varphi(-\infty)=0$, $\varphi(t+0)=\varphi(t)$) onto the class $\mathfrak{F}$ of Fourier-Stieltjes transforms. The continuity of the inverse mapping is discussed for various topologies on $\mathfrak{F}$ and $\mathfrak{M}$.

Let $V(\delta)$ denote the set of Φ such that $\Phi(0)<\delta$. The "almost weak" topology is that whose neighbourhoods of 0 are intersections of the neighbourhoods in the weak topology with $L(-\infty,\infty)$ as dual and a $V(\delta)$; the "mean value" that whose neighbourhoods are intersections of a $V(\delta)$ and a set $\lim (2T)^{-1}\int_{-T}^{T}|\Phi(x)|dx<\delta$; the "mean almost weak" is the upper bound of these two topologies.

Typical results are: the mapping of $\mathfrak{F}$ to $\mathfrak{M}$ is continuous with the topology of uniform convergence on $\mathfrak{M}$ everywhere if $\mathfrak{F}$ has the mean almost weak topology and at points which are transforms of continuous functions if $\mathfrak{F}$ has the almost weak topology. It is continuous if both spaces have the topology of uniform convergence. Results for the transformation of the space of functions of bounded variation are also given. *J. L. B. Cooper* (Cardiff).

★ **Janet, Maurice. Compléments divers sur la transformation de Laplace et les équations aux dérivées partielles.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955. 48 pp. 350 francs.

These notes of a three-month course given in 1952 are a clear exposition of some of the basic properties of Fourier integrals and one-dimensional Laplace transforms, and the solution of the heat equation $u_t-u_{x^2}=0$ and the wave equation $u_{t^2}-a^2(u_{x_1^2}+\cdots+u_{x_n^2})$ for the cases $n=1, 2, 3$. *D. L. Bernstein* (Rochester, N.Y.).

Garder, A. O., Jr. Functions completely monotonic with respect to a sequence. Proc. Amer. Math. Soc. 6 (1955), 919–928.

In their work on the convolution transform, Hirschman and Widder generalized the theorem due to S. Bernstein giving a necessary and sufficient condition that a function can be represented as a Laplace transform $\int_0^\infty e^{-xt}d\alpha(t)$, namely that the function be completely monotone on the half-line. Their necessary and sufficient condition that f be represented as a convolution transform $f(x)=\int_{-\infty}^{\infty} G(x-t)d\beta(t)$, where $G(t)$ is a certain totally positive kernel, is that $f(x)=o(e^{ax})$ and $g_k(x)=\prod_{n=1}^{k}(1-D/a_n)f(x)\geqq 0$. The present paper uses this last expression as a basis for a definition of a function being completely monotonic (D, A); the author then shows it is equivalent to another definition of complete monotonicity (Δ, A), just as Bernstein had shown his definition of complete monotonicity was equivalent to another expression in terms of finite differences. From this second definition, Bernstein had shown that if every term of a convergent sequence of functions is a Laplace transform, then the limit function is also. The present paper, using the (Δ, A) definition, proves that if every term of a convergent sequence of functions is representable as a convolution transform, the same is true about the limit function. *D. L. Bernstein* (Rochester, N. Y.).

Pistoia, Angelo. Sul prodotto di composizione, nella teoria della trasformata doppia di Laplace. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 18(87) (1954), 627–652.

The author considers both the double convolution of two functions $F*G(x,y)$ and the convolution about an axis, $H_a^*F(x,y)$, and proves by means of a succession of theorems conditions under which the double Laplace transforms of these convolution are given by $f(s,t)g(s,t)$ and $f(s,t)h(s\cos\alpha+t\sin\alpha)$, where f, g, h are the transforms of F, G, H. For example, a function is integrable of order $\alpha>-1$ in the first quadrant if $F*(xy)^\alpha/(xy)^\alpha$ is bounded and has a finite limit as x and y become infinite; then if both the functions $e^{-sx-ty}F(x,y)$ and $e^{-sx-ty}G(x,y)$ are integrable of orders >-1, the formula for the double convolution holds. The author seems unaware of a paper by Coon and Bernstein [Trans. Amer. Math. Soc. 74 (1953), 135–176; MR 14, 639] which gives theorems concerning these same operations. *D. L. Bernstein.*

Standish, Charles. A generalization of the Poisson transform. Proc. Amer. Math. Soc. 6 (1955), 942–949.

An inversion formula is obtained for the transform

$$f(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A(z-t)+B}{(z-t)^2+C^2}\,d\alpha(t)$$

which is an extension of the inversion for the Poisson transform given by Pollard [Trans. Amer. Math. Soc. 78 (1955), 541–550; MR 16, 818]. If $T_\delta f(x)$ is defined as in the review referred to and $h(x)=-Af_0^x f(u)du+B|C|^{-1}f(x)$, then it is shown that

$$(*)\qquad \lim_{t\to 1-}\int_0^\infty GT_{tC}(u)du$$
$$=\tfrac{1}{2}[\alpha(x+0)+\alpha(x-0)-\{\alpha(0+)+\alpha(0-)\}]-AGKx,$$

where $G=C^2(B^2+C^2A^2)^{-1}$ and

$$K=-2(A^2G\log 4)^{-1}\lim_{\lambda\to\infty}\int_\lambda^{2\lambda}\frac{-At+B}{C^2+t^2}\,d\beta(t),$$

$\beta(t)$ denoting the left-hand side of (*). *F. Goodspeed.*

Standish, Charles. On the representation of a function by a Poisson transform. Michigan Math. J. 3 (1955–56), 95–104.

The author gives necessary and sufficient conditions for

functions $f(x)$ to be representable in the form

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\alpha(y)}{1+(x-y)^2},$$

where $\alpha(y)$ is of bounded variation on $(-\infty, +\infty)$ or $\alpha(y)$ is nondecreasing and bounded on $(-\infty, +\infty)$, or in the form

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{g(y)dy}{1+(x-y)^2},$$

where $g(y)$ is integrable on $(-\infty, +\infty)$, or $g(y)$ is bounded on $(-\infty, +\infty)$, or $g(y)\in L_p$ on $(-\infty, +\infty)$ for some $p>1$. The conditions are phrased with the operators

$$(\cos tD)f(x),\quad \frac{\sin tD}{D}f(x) \text{ and } \hat{f}(x),$$

defined by H. Pollard [Trans. Amer. Math. Soc. **78** (1955), 541–550; MR **16**, 818]. *W. Saxer* (Zürich).

Special Functions

González, Mario O. Theory of elliptic functions. Rev. Soc. Cubana Ci. Fis. Mat. **3**, 67–75 (1954); 109–118 (1955). (Spanish)
Expository paper. *E. Grosswald* (Philadelphia, Pa.).

Mikolás, Miklós. Über ein Fejérsches Problem bezüglich unendlicher Produkte. Mat. Lapok **6** (1955), 198–209. (Hungarian. Russian and German summaries)
The author evaluates the infinite product

$$\prod_{n=0}^{\infty}\frac{[A+nD+(\nu-1)d/2]^{\nu}}{(A+nD)(A+nD+d)\cdots(A+nD+\nu d-d)}$$

in terms of gamma functions [see E. T. Whittaker and G. N. Watson, Modern analysis, 4th ed., Cambridge, 1927, § 12.13] and shows that an expression in terms of elementary functions results if either $\delta=d/D$ is an integer and $(\nu-1)\delta$ is even or $[2A+(\nu-1)d]/D$ is an integer other than $0, -2, -4, \cdots$. *A. Erdélyi.*

Colombo, Serge. Sur les fonctions $\nu(x, n)$ et $\mu(x, m, n)$. Bull. Sci. Math. (2) **79** (1955), 72–78.
The author presents a number of formulas to complement the material in section 18.3 of "Higher transcendental functions" [v. 3, McGraw-Hill, New York, 1955; MR **16**, 586] by the reviewer and others. In addition to giving several Mellin and Laplace transform formulas, he shows that $t^{-1/2}[e^t-\nu(t)-\frac{1}{2}]$ is skew-reciprocal in the Hankel transformation of order zero. *A. Erdélyi.*

Cooke, J. C. On the sums of certain series of Bessel functions. Monatsh. Math. **59** (1955), 253–257.
The author evaluates $\sum_{n=0}^{\infty}t^{2k+\nu}J_{2k+\nu}(z)$ as a contour integral, and gives transformations of this contour integral in several cases. [See also H. M. Macdonald, Electric waves, Cambridge, 1902, Appendix D.]
A. Erdélyi (Pasadena, Calif.).

Ragab, F. M. Some formulae for the product of three modified Bessel-functions of the second kind. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 621–626.
Evaluation of

$$\int_0^{\infty}e^{-t}t^{k-1}K_{\lambda}(t)f(t)dt,$$

where $f(t)$ is $K_m(xt)K_n(x, t)$, $K_m(e^{i\pi}xt)K_n(xt)$, $K_m(x/t)\cdot K_n(x/t)$, or $K_m(e^{i\pi}x/t)K_n(x/t)$. *A. Erdélyi.*

Janković, Zlatko. Über die Legendreschen Funktionen. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 3–36. (Serbo-Croatian summary)
The author reconstructs numerous formulas involving Legendre functions. He refrains from using the theory of hypergeometric series and bases his work as far as possible on recurrence relations. *A. Erdélyi.*

Henrici, Peter. Addition theorems for general Legendre and Gegenbauer functions. J. Rational Mech. Anal. **4** (1955), 983–1018.
Further consequences of a previous idea of the author are explored [MR **14**, 872; **15**, 710, 871; **17**, 262]. In particular, expansions of the class C of solutions of the elliptic equation

$$\nabla^2u+(2\mu+1)x^{-1}\partial u/\partial x+(2\nu+1)y^{-1}\partial u/\partial y=0,$$

regular at $x=c$, $y=0$, $c\neq 0$, are developed. This is accomplished by means of a continuous, linear one-to-one mapping of the class C onto the class of functions analytic at the origin. It follows by Taylor's theorem that every solution in C may be expanded in a series of "normal solutions" $u_n(x, y)$ corresponding to the functions z^n. These normal solutions are found to be certain combinations of Legendre functions and Gegenbauer polynomials (L.G. functions). The precise mathematical formulation of these statements is the expansion theorem established in Part I of the paper.
In Part II special cases of the expansion theorem are examined. There results a systematic account of the addition theorems for Legendre and Gegenbauer functions. This contains three new and very general expansions of an arbitrary Legendre or Gegenbauer functions in L.G. functions. Special and limiting cases are the classical addition theorems of the Legendre functions and Gegenbauer polynomials, Gegenbauer's and Graf's addition theorems of the Bessel functions and a generalization of Neumann's expansion for $(\xi-\zeta)^{-1}$ in terms of Legendre functions. Yet another generalization of Neumann's formula (the L.G. expansion of a certain ${}_2F_1$), a new series for the product of two Bessel functions and eight definite integrals, which may be specialized to a large number of known ones, are established. For the specific form of all these results the reader must be referred to the paper itself. *N. D. Kazarinoff* (Lafayette, Ind.).

★ **Campbell, Robert. Théorie générale de l'équation de Mathieu et de quelques autres équations différentielles de la mécanique.** Masson et Cie, Paris, 1955. xvi+271 pp.
In this book the author sets out to describe the various methods employed in the investigation of Mathieu and related functions, and compares the results achieved by the various methods. He takes the Mathieu equation in the form $y''+(p-2p\cos 2z)y=0$, and beside it he also discusses "Whittaker's equation" $y''+(a+2b\cos 2z+2c\cos 4z)y=0$, the "associated Mathieu equation" $y''-2\nu\tan z\,y'+(a+k^2\sin^2 z)y=0$ (which is closely related to the differential equation of spheroidal wave functions), and Lamé's equation $y''-[n(n+1)k^2\,\mathrm{sn}^2 z+h]y=0$.
Contents: Introduction. Relations between the above differential equations and other differential equations of mathematical physics, and occurrence of the equations in mathematical physics. Part I. Mathieu functions. Periodic

solutions, Fourier expansions, characteristic values, Bessel function expansions, integral equations, asymptotic solutions for large values of the parameter. Part II. Functions of the second kind. Definition and various representations. Part III. General theory of the Mathieu equation. Floquet's theory, characteristic exponent, stable and unstable solutions, asymptotic form of the general solution, various algebraic forms of Mathieu's equation and the Klein-Bôcher classification of differential equations of the second order with rational coefficients.

The book is well equipped with graphs and lists of formulas and with a ten-page bibliography.

A. Erdélyi (Pasadena, Calif.).

Ou Tchen-Yang, Vincent. Contribution à l'étude des functions hypergéométriques de deux variables. Ann. Sci. Ecole Norm. Sup. (3) **72** (1955), 1–68.

Riemann's characterization of Gauss' hypergeometric function by means of the singularities, and the exponents belonging to these singularities, of the hypergeometric differential equation, has been extended by Picard to Appell's hypergeometric function of two variables, F_1. The author sets himself the task to extend similarly Schwarz' geometrical characterization of the hypergeometric function.

In Schwarz' work the real axis plays an important role. It is the circle carrying all singularities, and it divides the complex plane in two half-planes each of which is mapped on a triangle by a quotient of two solutions of the hypergeometric equation. In the case of two complex variables x and y, the author correspondingly takes the hypercones $\mathrm{Im}\, x=0$, $\mathrm{Im}\, y=0$, $\mathrm{Im}\,(y/x)=0$, $\mathrm{Im}\,[(1-y)/(1-x)]=0$ which carry all singularities of the system of partial differential equations satisfied by F_1. The four hypercones divide the 4-dimensional space in 14 regions. The author investigates the nature of these regions, selects seven to form the "fundamental half-domain" (the other seven arising when x and y are replaced by their complex conjugates). He then obtains the ordinary differential equation satisfied by F_1 and F_2 when x and y depend on a single parameter t, with a view of investigating the analytic continuation along closed curves and hence obtaining the monodromy group of F_1.

Next Picard's contour integrals for the solutions of the system of F_1 are discussed at great length. [This discussion does not yield many new results, and fails to take into account some of the important contours; see Erdélyi, Acta Math. **83** (1950), 131–164; MR **12**, 257.] From here the author proceeds to canonical fundamental systems for a transformation of the monodromy group and, for real parameters of F_1, he constructs the regions corresponding to Schwarz' triangles, and studies the dependence of their "angles" on the parameters of F_1. The paper concludes with a study of these regions. *A. Erdélyi.*

Čaĭkovskiĭ, Ya., and Titc, T. On the real zeros of the confluent hypergeometric series. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 161–165. (Russian)

The author constructs a Sturmian chain of suitable multiples of ${}_1F_1(a+n; c+n; x)$, $n=0, 1, 2, \cdots$, and uses it to establish the known distribution [A. Erdélyi et al., Higher transcendental functions, vol. 1, McGraw-Hill, New York, 1953, p. 289; MR **15**, 419] of real zeros of ${}_1F_1(a; c; x)$ when a and c are real. *A. Erdélyi.*

See also: Carlitz, p. 460; Miles and Williams, p. 474; Bihari, p. 489; Kreyszig, p. 492; Alder and Winther, p. 566.

Ordinary Differential Equations

Reĭzin, L. E. The behavior of the integral curves of a system of three differential equations in the neighborhood of a singular point. Amer. Math. Soc. Transl. (2) **1** (1955), 239–252.

Translated from Latvijas PSR Zinātņu Akad. Vēstis **1951**, no. 2(43), 333–346; MR **15**, 311.

Erugin, N. P. Remark on the paper of N. P. Erugin, "On extensions of solutions of differential equations", Prikl. Mat. Meh., v. 15, no. 1, 1951. Prikl. Mat. Meh. **19** (1955), 764. (Russian)

Bibliographical remarks concerning his earlier paper [same journal **15** (1951), 55–58; MR **12**, 611], in particular, calling attention to a paper by Mayerhofer [Monatsh. Math. Phys. **41** (1934), 183–187].

Coles, W. J. Linear and Riccati systems. Duke Math. J. **22** (1955), 333–338.

This paper is concerned with an ordinary linear vector differential equation

$$(*) \qquad x'=A(t)x,\ A(t)\equiv\|a_{ij}(t)\|\ (i,j=1,\cdots,n),$$

and associated Riccati systems of the form

$$(**) \qquad y_i'=-y_i\sum_{k=1}^{n-1}a_{nk}y_k+\sum_{k=1}^{n-1}(a_{ik}-\delta_{ik}a_{nn})y_k+a_{in} \qquad (i=1,\cdots,n-1);$$

in particular, (**) is satisfied by $y_i=x_i/x_n$ $(i=1,\cdots,n-1)$, where $x\equiv(x_j(t))$ $(j=1,\cdots,n)$, is a solution of (*). Results on the equivalence of (*) and (**) are given, and there is presented a general solution of (**) in terms of $n+1$ particular solutions of this equation. Finally, it is shown how the knowledge of n solutions of (**) leads to a matrix $P(t)$ and a diagonal matrix $B(t)$ such that (*) is equivalent to $y'=B(t)y$ under the transformation $y=P(t)x$. The author notes relations between the derived results and those of Chiellini [Rend. Sem. Fac. Sci. Univ. Cagliari **18** (1948), 44–58; MR **11**, 665] on Riccati systems.

W. T. Reid (Evanston, Ill.).

Fettis, H. E. Particular integrals of linear differential equations. Amer. Math. Monthly **62** (1955), 174–176.

The author shows that $Y(x)=\int_0^x G(x-\xi)f(\xi)\,d\xi$ solves the equation $L(Y)\equiv(D^n+a_1D^{n-1}+\cdots+a_n)Y=f(x)$ provided $L(G)=0$ and $G(0)=G'(0)=\cdots=G^{(n-2)}(0)=0$, $G^{(n-1)}(0)=1$. Similarly for systems. Of course, this is known to hold even when the a_k are functions of x, in which case $G(x-\xi)$ is to be replaced by $G(x,\xi)$, and $x=0$ by $x=\xi$ in the initial conditions.

M. Golomb (Lafayette, Ind.).

Gel'fand, I. M., and Lidskiĭ, V. B. On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1(63), 3–40. (Russian)

The system, of $2k$ equations, is equivalent to the matrix differential equation $dY/dt=IH(t)Y$, where $H(t)$ is real, symmetric, piecewise continuous and of period ω in t $(-\infty<t<\infty)$, and I has the form $\begin{pmatrix}0 & E_k\\ -E_k & 0\end{pmatrix}$, E_k being the kth order unit matrix. The system is to be "strongly stable" in the sense that all solutions of $dY/dt=I\tilde{H}(t)Y$ are bounded as $t\to\infty$ for any $\tilde{H}(t)$ of the same form subject to $\|\tilde{H}(t)-H(t)\|<\varepsilon$ for some $\varepsilon>0$. The

problem is to enumerate the connected regions formed by the set of such $H(t)$, within one of which any such $H(t)$ can be continuously deformed into any other. Fixing $Y(t)$ by $Y(0)=E_{2k}$, $dY/dt=IH(t)Y$, it is observed that $Y(t)$ $(0\leq t\leq\omega)$ traces a curve from E_{2k} to $Y(\omega)$ within the group of real symplectic matrices, and that any such curve, if suitably smooth, corresponds to an $H(t)$; furthermore, the strong stability depends only on $Y(\omega)$. This reduces the problem to (i) the enumeration of the connected regions formed by those $Y(\omega)$ which satisfy the strong stability conditions, and (ii) the determination of the connectivity of the group of real symplectic matrices. The solution of (i) is based on results announced by M. G. Kreĭn [Dokl. Akad. Nauk SSSR (N.S.) **73** (1950), 445–448; MR **12**, 100], proofs being now supplied; there are 2^k regions for $Y(\omega)$, according to whether the k characteristic numbers of $Y(\omega)$ which lie on the upper half of the unit circle are of the first or second kinds, i.e. move into or outside the unit circle when $dY/dt=IH(t)Y$ is perturbed to $dY/dt=I(H(t)+\lambda Q(t))Y$, where λ is small, $\operatorname{Im}\lambda>0$, and $Q(t)$ is periodic, real, symmetric and positive-definite. Concerning (ii), it is proved that the group of real symplectic matrices is homomorphic to the topological product of the circumference of a circle and a simply-connected topological space; this is deduced from the fact that the set of unitary unimodular matrices form such a space [see, e.g., H. Weyl, The classical groups, Princeton, 1939, p. 268; MR **1**, 42]. The final result is that there are 2^k infinite families of regions of the $H(t)$, each member of a family being fixed by an integral rotation-number $n(-\infty<n<\infty)$. There is a special section on the case $k=1$, already treated by V. A. Yakubovič [see the second following review]. *F. V. Atkinson.*

Lidskiĭ, V. B. Oscillation theorems for canonical systems of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 877–880. (Russian)

Let the $2k$th-order matrices

$$I=\begin{pmatrix}0 & E_k\\ -E_k & 0\end{pmatrix},\quad Y(t)=\begin{pmatrix}y_1(t) & y_2(t)\\ y_3(t) & y_4(t)\end{pmatrix},\quad H(t)=\begin{pmatrix}h_1(t) & h_2(t)\\ h_3(t) & h_4(t)\end{pmatrix},$$

where $H(t)$ is real and symmetric, E_k is the unit kth-order matrix, and $y_s(t)$, $h_s(t)$ are kth-order matrices, satisfy the differential equation $dY/dt=IH(t)Y$, with $Y(0)=E_{2k}$. The "oscillation theorems" deal with the characteristic values $\varrho_1(t),\cdots,\varrho_k(t)$ of the (unitary and symmetric) matrix $(y_1(t)-iy_2(t))^{-1}(y_1(t)+iy_2(t))$. If $h_4(t)$ is positive-definite, then as t increase the $\varrho_s(t)$ move round the unit circle in the positive sense. Among other results is a comparison theorem for $dY_1/dt=IH_1(t)Y_1$, $dY_2/dt=IH_2(t)Y_2$ when $H_1(t)>H_2(t)$. With an appropriate numeration we have $\arg\varrho_s^{(1)}(t)>\arg\varrho_s^{(2)}(t)$ for all $t>0$. Separation theorems for the zeros of the determinants of $y_1(t)$, $y_2(t)$ are also mentioned. Slightly less general results had been given by M. G. Kreĭn [see the preceding review], and M. Morse [Math. Ann **103** (1930), 52–69].

F. V. Atkinson (Canberra).

Yakubovič, V. A. Questions of the stability of solutions of a system of two linear differential equations of canonical form with periodic coefficients. Mat. Sb. N.S. **37(79)** (1955), 21–68. (Russian)

The author gives proofs and a detailed account of results already announced by him [Dokl. Akad. Nauk SSSR (N.S.) **78** (1951), 221–224; MR **13**, 237]. The first half of the paper deals essentially with the case $k=1$ of the problem considered by Gel'fand and Lidskiĭ (second preceding review, which see for notation). The present treatment uses different methods, and is considerably more detailed; also Lebesgue integrability is required of $H(t)$ in place of piecewise continuity. A long footnote (pp. 42–43) suggests a method of extending the argument beyond the case $k=1$. The author next finds conditions on $H(t)$ in order that the system should belong to a particular region of stability or instability; criteria are deduced regarding the stability of $y''+p(t)y=0$, where $p(t)$ is periodic. A section deals with a "comparison theorem", similar to that of Lidskiĭ (see preceding review), and its consequences. For example, if matrix functions $H_1(t)$, $H_2(t)$ give systems belonging to the same stability region, then so does any $H(t)$ with $H_1(t)\leq H(t)\leq H_2(t)$; geometrical interpretations are given, and further stability criteria for $y''+p(t)y=0$. After a section apparently intended for numerical application, the author concludes with a section on the "oscillation theorem". He gives conditions, too involved to reproduce, for the stability and instability zones on the real λ-axis arising from the parametric case $H(t)=H_1(t)$ to be separated by the eigen-values of the periodic and semi-periodic boundary conditions, occurring in a certain order. A particular case is $H_\lambda(t)=\lambda H_1(t)+H_2(t)$, where $H_1(t)\geq 0$ and $H_1(t)>0$ on a set of positive measure. *F. V. Atkinson.*

Taam, Choy-Tak. Schlicht functions and linear differential equations of second order. J. Rational Mech. Anal. **4** (1955), 467–480.

A differential equation (*) $w''(z)+p(z)w(z)=0$ is said to be disconjugate in a region D if no solution of (*) vanishes in D more than once. If D is a domain in the complex z-plane, disconjugacy of (*) in D is equivalent to the univalence in D of the ratio of two linearly independent solutions of (*), and criteria of disconjugacy will therefore amount to criteria of univalence for functions $f(z)$ in terms of the Schwarzian derivative $p(z)=\{f(z),z\}$. In the present paper, the author discusses a number of such criteria. The main part of the paper is devoted to reproving some criteria of this type obtained previously by the reviewer [Bull. Amer. Math. Soc. **55** (1949), 545–551; Lectures on functions of a complex variable, Univ. of Michigan Press, 1955, pp. 49–60; MR **10**, 696; **16**, 1093] by means of a slightly different method. *Z. Nehari.*

Erugin, N. P. Certain general problems of the qualitative and analytic theory of linear systems of differential equations. Prikl. Mat. Meh. **19** (1955), 211–221. (Russian)

This is a lucidly written exposition of some of the published results in the theory of linear differential equations (1) $dX(x)/dz=XP(z)$ (matrices X, P of n^2 elements). When P is in the form of a Laurent series near $z=a$, then the solution (normed, say, at a point $b\neq a$) is $X(z)=(z-a)^WZ(z)$, where W is a constant matrix and Z is uniform around a (is a Laurent series). The cases considered are those when a is a regular singular point, when (1) is of Fuchsian type, and when (1) has at, say, $z=0$ an irregular singular point. The Poincaré-Lappo Danilevsky problem is that of explicit determination of W [Poincaré, Acta Math. **4** (1883), 209–312; Lappo Danilevsky, Trudy Fiz.-Mat. Inst. Steklov **6** (1934)]. A number of author's developments [Erugin, Mat. Sb. **42** (1935), 745–753; N.S. **3(45)** (1938), 509–526; Trudy Mat. Inst. Steklov **13** (1946); MR **9**, 509] form part of the exposition. [The latter of these is perhaps the best work on reducible systems.] When $z=t$ is real and $P(t)$ is period-

ic, the solution is $X(t)=e^{At}Z(t)$, where A is constant and $Z(t)$ is periodic. The fundamental problem (connected with that of reducibility) of computation of A and $Z(t)$ is solved. A connection is established with the relevant formulas of Lappo Danilevsky. Effective applications to the Mathieu equation are given. According to K. P. Persidsky [Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. 1 (1947)] the so called correct systems have the property that their characteristic numbers are limits of the characteristic numbers of suitable sequences of periodic systems; whence the results described in the review are applicable to correct systems.

W. J. Trjitzinsky (Urbana, Ill.).

Redheffer, R. M., and Wasow, W. On the convergence of asymptotic solutions of linear differential equations. Pacific J. Math. 5 (1955), 817–834.

The authors consider singular perturbation problems of the form

$$y^{(n)}+\varepsilon^{-h}\sum_{1}^{n}A_j(x,\varepsilon)y^{(n-j)}=0$$

with $A_j(x,\varepsilon)$ analytic in x and ε for some range of x, and near $\varepsilon=0$. They investigate the possibility of there being solutions of such equations which are themselves analytic in ε about $\varepsilon=0$. A number of examples are given to the affirmative, but none of the conditions obtained is sufficient to guarantee their existence in general. *G. Latta.*

Hartman, Philip, and Wintner, Aurel. On an oscillation criterion of de la Valleé Poussin. Quart. Appl. Math. 13 (1955), 330–332.

A criterion of de la Vallee Poussin, when particularized for a second-order equation, is improved considerably. Let $x''+g(t)x'+f(t)x=0$, where g and t are continuous, have a non-trivial solution vanishing at $t=0$ and $t=h$. Then

$$h<\max\{\textstyle\int_0^h t|g(t)|dt,\ \int_0^h (h-t)|g(t)|dt\}+\int_0^h t(h-t)|f(t)|dt.$$

The short proof uses a device due to Nehari.

N. Levinson (Cambridge, Mass.).

Staržinskiĭ, V. M. A survey of works on the conditions of stability of the trivial solution of a system of linear differential equations with periodic coefficients. Amer. Math. Soc. Transl. (2) 1 (1955), 189–237.

Translated from Prikl. Mat. Meh. 18 (1954), 469–510; MR 16, 249.

★**Krein, M. G. On the inverse problem of filter theory and the λ zone of stability.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 7 pp. $3.50.

Translated from Doklady Akad. Nauk SSSR (N.S.) 93 (1953), 767–770; MR 15, 874.

Krein, M. G. On certain problems on the maximum and minimum of characteristic values and on the Lyapunov zones of stability. Amer. Math. Soc. Transl. (2) 1 (1955), 163–187.

Translated from Prikl. Mat. Meh. 15 (1951), 323–348; MR 13, 348.

★**Davis, Harold T. Studies relating to a non-linear differential equation of second order, with special reference to the first and second transcendents of Painlevé.** Non-linear differential equations of the second order, pp. 1–72. OOR Project No. 956, technical report. Northwestern University, Evanston, Ill., 1955.

The object of this investigation is the study of certain differential equations of the form

(1) $\quad Ay''+By'+Cy'^2+D=0,\ y'=dy/dx$, etc.,

where $A, \cdots, D$ are polynomials in y with coefficients which are functions of x [see also E. L. Ince, Ordinary differential equations, Longmans-Green, London, 1926, p. 317 ff.]. Many differential equations occurring in applied mathematics are of the form (1), and so are some important classes of differential equations. Among the latter are equations whose critical points are fixed, and which can be reduced to one of fifty canonical types. Of these all but six were found to be integrable in terms of "known" functions. The first two of the remaining six are the differential equations for Painlevé's first and second transcendents,

(2) $\quad y''=6y^2+\lambda x,$ $\qquad$ (3) $\quad y''=2y^3+xy+\mu,$

and these are studied in detail in the present report.

First the author investigates the effect of a linear fractional transformation of y on (1), singling out those transformations not increasing the degree of the polynomials $A, \cdots, D$. Then he discusses transformations, in particular linear fractional transformations, of x.

The major part of the paper is concerned with (2) and (3). If $\lambda=0$, the solution of (2) can be expressed as

$$y(x,0)=-\frac{C^2k^2}{1+k^2}+\frac{C^2}{\operatorname{sn}(C(x-x_1),k)},$$

where x_1 and C are arbitrary constants while $1-k^2+k^4=0$. For general λ, the author writes

$$y(x,\lambda)=A+\frac{B}{S^2(C(x-x_1))},$$

where x_1 and C are arbitrary, A and B depend on λ, C, and x_1, and $S(v)$ is a new transcendental function which reduces to $\operatorname{sn} v$ when $\lambda=0$. Various expansions are obtained resulting in the expansions of $y(x,\lambda)$ (i) around a pole, (ii) around an ordinary point. These are used for the tabulation of the first Painlevé transcendent. Tables of y and y' (initial values: $y=1$, $y'=0$ at $x=0$) are given for $\lambda=0(1)5$, $x=-1(.01)1$, and there are also some graphs.

For the solutions of (3) there are also expansions valid around a singular point and a regular point respectively, tables of y and y' (initial values: $y=1$, $y'=0$ at $x=0$) for $\mu=0(1)5$ and $-1(.01)1$ (shorter range for $\mu\geqq 3$), and graphs.

The paper concludes with four appendices giving lists of formulas, and a bibliography of 41 items. *A. Erdélyi.*

Peyovitch, T. Sur un théorème des équations différentielles algébriques. Bull. Soc. Math. Phys. Serbie 6 (1954), 74–79. (Serbo-Croatian. French summary)

This paper contains a simplified proof for the following theorem due to M. Petrovitch [Publ. Math. Univ. Belgrade 6–7 (1938), 290–325]: Every integral of the algebraic differential equation $f(x,y,y',\cdots,y^{(p)})=0$ can be obtained in the form $x=e^{\phi_1(t)}$, $y=e^{\phi_2(t)}$, where $\phi_1(t)$, $\phi_2(t)$ are linear combinations of the functions $\int_0^t e^{u_k(s)}ds$ and the $u_k(t)$ are defined by a system

$$\frac{du_i}{dt}=\sum_{k=1}^{m}a_{ik}e^{u_k}\ (i=1,2,\cdots,m)$$

with constant coefficients a_{ik}. *M. Golomb.*

Hartman, Philip, and Wintner, Aurel. Asymptotic integrations of ordinary non-linear differential equations. Amer. J. Math. 77 (1955), 692–724.

Non-linear systems are considered of the form

$$(*) \qquad y'=JY+F(t, y),\ y=(y', \cdots, y^n)\ (t\geqq t_0),$$

where J is a constant matrix and F is a continuous vector function for $t\geqq t_0$ and $|y|\leqq$ const., with $|F(t, y)|/|y|\to 0$ as $(t, y)\to(\infty, 0)$. No Lipschitz condition is required so that no local uniqueness is assumed. Asymptotic theorems are given which generalize previous ones of Poincaré for F analytic, of O. Perron for F non-analytic under a Lipschitz condition, and of the same authors and of O. Dunkel for the linear case. There is no loss in generality in supposing that J has a normal form, i.e., (*) has the form

$$dy^{j_1}/dt=\lambda y^{j_1}+F_{j_1},\ dy^{j_k}/dt=\lambda y^{j_k}+y^{j_{k-1}}+F_{j_k}\ (k=2, \cdots, h),$$

where $\lambda=\lambda(j)$, $h=h(j)$, $j=1, \cdots, g$, $h(1)+\cdots+h(g)=n$. If $\mu(j)=\Re\lambda(j)$, let $\mu^1<\mu^2<\cdots<\mu^f$ be the distinct numbers $\mu(j)$; for every $1\leqq m\leqq f$ let q, p denote all integers with $\mu(q)=\mu^m$, $\mu(p)<\mu^m$; for every q, or p, let k denote all integers $1\leqq k\leqq h(q)$, or $h(p)$. Finally let $L_m=\sum_q\sum_k |y^{qk}|^2$. (I) If $\mu=\mu^m<0$, then, given $0<\eta<1$, there exist $\delta=\delta(\eta)>0$, $T=T(\eta)>0$, such that for all $t_0\geqq T$ and set of $\sum_p h(p)+\sum_q h(q)=N<n$ numbers y_0^{pk}, y_0^{qk} with $\sum_p\sum_k |y_0^{pk}|^2<\eta\sum_q\sum_k |y^{qk}|^2<\eta\delta$, there exists a solution $y=y(t)$, $t\geqq t_0$, of (*) satisfying the partial set of initial conditions $y^{pk}(t_0)=y_0^{pk}$, $y^{qk}(t_0)=y_0^{qk}$, and $L_j=o(L_m)$ if $j<m$, $\log|y(t)|=(\mu^m+o(1))t$ as $t\to+\infty$. This asymptotic result on the "logarithmic scale" is then refined into the following one of actual asymptotic evaluation of a convenient integral $y(t)$. Let $h^*=\max h(q)$, $h_0\geqq h_*$, $0\leqq j_0\leqq h_*$; Let $l(q)$, $k(q)$ be the least and the greatest integers such that $1\leqq l(q)\leqq k(q)\leqq \min[h(q), h_0-j_0]$, $h(q)-l(q)\geqq j_0$; let $y_{qk}(t)$ denote functions with $y_{qk}(t)=o(e^{\mu^q t}t^{k-b})$ for $1\leqq k\leqq k_0$, $y_{qk}(t)=e^{\lambda(q)t}t^{k-k_0}/(k-k_0)!+o(e^{\mu^q t}t^{k-b})$ for $k_0\leqq k\leqq h(q)$. (II) If $|F(t, y)|\leqq\phi(t, |y|)$ for $0\leqq t<+\infty$, $|y|\leqq\alpha_0$, $\alpha_0>0$, where ϕ is a scalar function non-decreasing in $|y|$ for every t and $\int^\infty e^{-\mu^q t}t^{h_0-j_0-1}\phi(t, \alpha e^{\mu^q t}t^{j_0})dt<+\infty$ for every $0<\alpha\leqq\alpha_0$, then, given arbitrary constants C^{qk} not all zero, there exists a solution $y=y(t)$ of the form $y=\sum_q\sum_k C^{qk}y_{qk}(t)$ where $\sum_k$ ranges over all $l(q)\leqq k\leqq k(q)$. Both statements (I) and (II) are shown to have partial converses and some more stringent forms. *L. Cesari.*

Łojasiewicz, S. Sur un effet asymptotique dans les équations différentielles dont les seconds membres contiennent des termes périodiques de pulsation et d'amplitude tendant à l'infini. Ann. Polon. Math. 1 (1955), 388–413.

It is known that the vertical position of a simple pendulum whose point of support is in simple harmonic motion with amplitude a and frequency ω is stable for sufficiently large ω and small a. In the context of a far more general problem the motion of the pendulum is considered when $\omega\to\infty$ while $a\omega$ remains constant. Suppose in the system

$$(I) \qquad \ddot{x}_i=f_i(t, x_j, \dot{x}_k)+\varphi_i(t, t, x_j)\ (i, j, k=1, 2, \cdots, n)$$

$\varphi_i(u, s, x_j)$ is periodic in s with period τ and with mean value zero. Define

$$\Phi_i(t, s, x_j)=\int_0^s\varphi_i(t, \sigma, x_j)d\sigma-\tau^{-1}\int_0^\tau\int_\sigma^u\varphi_i(t, \sigma, x_j)d\sigma du.$$

Then under suitable differentiability conditions upon f_i and φ_i for each suitable solution $\bar{x}_i$ of the system

$$(II) \qquad \ddot{x}_i=\tau^{-1}\int_0^\tau\Big[f_i(t, x_j, \dot{x}_k+\Phi_k(t, s, x_j))-\sum_1^n\partial\Phi_i(t, s, x_j)/\partial x_\nu\Phi_\nu(t, s, x_j)\Big]ds$$

there exists a solution of (I) uniformly convergent in t, $\alpha\leqq t\leqq\beta$, to $\bar{x}_i$ as $\tau\to 0$. In the case of the Hill equation $\ddot{x}=xp(t/\tau)$ equation (II) is $\ddot{x}=-\mu^2x$, where $\mu^2=\int_0^1 p^2ds-(\int_0^1 Pdx)^2$ and $P(x)$ is any primitive of $p(x)$. Other examples are discussed. The main theorem is generalized and restated in terms of distributions so that periodic impulse forcing terms may be handled. For example, the equation of motion of a particle acted upon by an impulse force

$$I(x)\lambda(t)=(kx+1)\sum_{-\infty}^{+\infty}(-1)^\nu\delta(t-\nu\tau/2)$$

of period $\tau/2$ is $\ddot{x}=\lambda(t)I(x)$. Here all functions and differentiation must be taken in the sense of distributions. The theorem shows that as $\tau\to 0$, solutions of $\ddot{x}=\lambda(t)I(x)$ converge to those of $4\ddot{x}=-kI(x)$. *N. D. Kazarinoff.*

Barbălat, I. L'allure globale des solutions de certaines équations différentielles non linéaires de second ordre. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 653–666. (Romanian. Russian and French summaries)

This paper extends the reviewer's results on the complete phase portrait of the system of two equations equivalent to van der Pol's equation [Contributions to the theory of nonlinear oscillations, v. 2, Princeton, 1952, pp. 61–73; MR 14, 557] to

$$(1) \qquad \ddot{x}+p(x)\dot{x}+x=0$$

and the equivalent system

$$(2) \qquad \dot{x}=y-P(x),\ \dot{y}=-x,\ p(x)=\sum_{i=-p}^{2n}d_ix^i,\ P(x)=\int_0^x p(\xi)d\xi,\quad 0\leqq p<2n.$$

Theorem 1. If p is even let $a_p<0$, if p is odd let the first coefficient a_{2i}, $i>p\neq 0$, be <0. Then there exists a stable limit cycle Γ, an innermost limit cycle Γ_i and an outermost limit cycle Γ_e. Γ_i is interior-stable, Γ_e is exterior-stable. Between the limit-cycles or between Γ_i and the origin the trajectories spiral in the usual manner. Outside Γ_e the behavior is as for van der Pol's equation. Theorem 2. Let

$$p(x)=\sum_{i=s}^n a_{2i}x^{2i},\ 0\leqq s<n,\ a_{2s}<0,\ a_{2n}>0,$$

and suppose that among the real roots of

$$\sum a_{2i}(2i+1)^{-1}x^{2(i-s)}=0$$

only one is of odd multiplicity. Then the global phase-portrait is the same as for van der Pol's equation. Theorem 3. Let

$$p(x)=\sum_{i=-p}^{2n}a_ix^i,\ 0\leqq p<n,\ n\geqq 1,\ a_{2n}>0.$$

(a) If p is odd let a_{p+2k+1} be the next odd coefficient, and suppose that $\sum_{i=k}^n a_{2i}(2i+1)^{-1}x^{2(i-k)}=0$, $k=\frac{1}{2}(p+1)+k$, possesses no real root of odd multiplicity. (b) If $p=2s$ let $a_{2s}>0$ for $s>0$, or else $0<a_0<2$ for $s=0$ and suppose the same as just stated for the real roots of

$$\sum_{i=s}^n a_{2i}(2i+1)^{-1}x^{2(i-s)}=0.$$

Then the origin is a positively stable focus for all

trajectories, and for $t\to-\infty$ the behavior is the same as for van der Pol's equation in the infinite region.

S. Lefschetz (Mexico, D.F.).

★ **Shimizu, Tatsujiro. On differential equations for oscillations with after effect and non linearity.** Proceedings of the Second Japan National Congress for Applied Mechanics, 1952, pp. 305–306. Science Council of Japan, Tokyo, 1953.

The author discusses integro-differential equations of which the equation

$$\ddot{x}+2\varepsilon\dot{x}+cx+dx^3-f\int_{-\infty}^{t}\varepsilon^{-\mu(t-s)}x(s)ds=E\sin\omega t$$

is typical. It is stated that for a given $\varepsilon>0$ this equation has a periodic solution if f is sufficiently small and μ sufficiently large. Little else of a specific nature is stated and nothing is proven. *C. E. Langenhop* (Ames, Ia.).

Arnold, F. R. Steady-state behavior of systems provided with nonlinear dynamic vibration absorbers. J. Appl. Mech. 22 (1955), 487–492.

A vibrating system with nonlinear dynamic coupling, subject to sinusoidal excitations is analyzed. The Ritz averaging procedure is used, and the system behavior is described by the usual response diagrams showing amplitude of response as a function of frequency.

W. S. Loud (Cambridge, Mass.).

Klotter, Karl. Free oscillations of systems having quadratic damping and arbitrary restoring forces. J. Appl. Mech. 22 (1955), 493–499.

The differential equation (*) $\ddot{q}+\frac{1}{2}\phi(q)\dot{q}^2+\frac{1}{2}\psi(q)=0$ may be transformed by the substitution $V=\dot{q}^2$ to the linear first-order equation (**) $dV/dq+\phi(q)V+\psi(q)=0$. Several particular cases of (*) are studied with the aid of the explicit solution of (**). The solution of (**) leads to exact relationships between two consecutive maximum displacements (extrema of q). Examples are worked out numerically and illustrated by graphs. *W. S. Loud.*

Wasow, Wolfgang. On the convergence of an approximation method of M. J. Lighthill. J. Rational Mech. Anal. 4 (1955), 751–767.

The author illustrates Lighthill's perturbation technique for obtaining approximate solutions of some nonlinear ordinary differential equations near their singular points [Phil. Mag (7) 40 (1949), 1179–1201; MR 11, 518], giving the proof of the convergence in a simple, but typical example:

$$(x+\alpha u)\frac{du}{dx}+q(x)u-r(x)=0;\ u(1)=b;\ \alpha>0,\ b>0.$$

The essential feature of the expansion procedure is to expand both x and u in powers of the parameters α as follows

$$x=\xi+\alpha x_1(\xi)+\alpha^2x_2(\xi)+\cdots,$$
$$u=u_0(\xi)+\alpha u_1(\xi)+\cdots+\alpha^m u_m(\xi)\ (0\leqq m),$$

and determine the coefficients recursively. The series is shown to converge for

$$|\alpha|<\gamma\xi^{\tau}\ (0\leqq\xi\leqq1),$$

where $\gamma>0$ is constant, and τ is a constant depending on $q(0)$. It is pointed out that nontrivial modifications of this procedure are necessary to handle examples such as

$$(x+\alpha u)\frac{du}{dx}+q(x)u=r(x)+\alpha u^3 \text{ or}$$

$$(x+\alpha u^2)\frac{du}{dx}+q(x)u=r(x),$$

for the same type of boundary condition. *G. E. Latta.*

Popov, E. P. Approximate investigation of auto-oscillations and forced oscillations of nonlinear systems of high order on the basis of harmonic linearization of nonlinearities. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 5, 3–38. (Russian)

The author applies the Krylov-Bogolyubov method of harmonic balance to the investigation of auto-oscillations and forced oscillations of various nonlinear systems. A typical example is: The system with two degrees of freedom, described by the equations

$$(1)\qquad (b_0p^2+b_2)x_2+F(px_2)=(k_1+k_2p)x_1,$$
$$(2)\qquad Q_2(p)x_1+R_2(p)x_2=0,$$

where $p=d/dt$, Q_2, R_2 are polynomials with constant coefficients and F is an arbitrary nonlinear function, is assumed to have a solution of the form $px_2\approx a\sin\omega t$ with unknown a, ω. Then $F(px_2)$ is replaced by $q(a)px_2$, where $q(a)=(\pi a)^{-1}\int_0^{2\pi}F(a\sin u)\sin u\,du$ and (1) becomes (1′) $Q_1(p)x_1+R_1(p)x_2=0$. a and ω are determined from $X=\mathrm{Re}\,D(i\omega)=0$, $Y=\mathrm{Im}\,D(i\omega)=0$, where $D=Q_2R_1-R_2Q_1$. To establish the stability of such a solution the author proposes two methods. One, to examine the harmonically linearized system (1′), (2) with the periodic coefficients replaced by their time averages. Two, to postulate that the system has solutions

$$x_2\approx(a+\Delta a)e^{-\alpha t}\sin(\omega+\Delta\omega)t$$

with $\alpha>0$ for all sufficiently small Δa and $\Delta\omega$, which leads to the condition

$$(3)\qquad \frac{\partial X}{\partial a}\frac{\partial Y}{\partial\omega}-\frac{\partial Y}{\partial a}\frac{\partial X}{\partial\omega}>0.$$

A number of systems are treated in detail by these methods. There is no discussion concerning the validity of the approximations made. The author does not seem to restrict his method to systems whose nonlinearities represent small perturbations. *M. Golomb.*

Aĭzerman, M. A., and Smirnova, I. M. Remarks on the paper of E. P. Popov, "Approximate investigation of auto-oscillations and forced oscillations of nonlinear systems of high order on the basis of harmonic linearization of nonlinearities". (Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1954, no. 5, 3-38 (1954).) Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 10, 185–189. (Russian)

The authors criticize the method of harmonic balance as applied by Popov [see preceding review] and others to systems where the nonlinear terms do not represent small perturbations. Although in such cases this method may give acceptable results for the amplitude and frequency of existing auto-oscillations or forced oscillations (the reason being that the linear part acts as a filter for the harmonics generated by the nonlinear components) it may lead to entirely erroneous results when applied to questions of stability. Using a simple linear system with constant

coefficients the authors show that inequality (3) of Popov's paper is neither necessary nor sufficient for stability. That the harmonically linearized system with averaged coefficients cannot be used either for determining stability is shown by the Mathieu equation.
M. Golomb (Lafayette, Ind.).

Popov, E. P. Answer to the remarks of M. A. Aĭzerman and I. M. Smirnova on the paper of E. P. Popov. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 10, 189–191. (Russian)
The author agrees with the critics of his paper [see preceding two reviews] that his condition (3) cannot be used to establish the stability of forced periodic oscillations. However, he claims it will give correct results for the stability of auto-oscillations of properly restricted systems. *M. Golomb* (Lafayette, Ind.).

Popov, E. P. Improvement of the first approximation for the investigation of auto-oscillations of nonlinear systems of high order. Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 345–348. (Russian)
Referring to systems as considered in a preceding paper by the author [reviewed third above] assume that in calculating the first harmonic of the nonlinear term $F(x, px)$ ("harmonic linearization") it is necessary to use some of the higher harmonics of its arguments. The correction terms for the amplitude and frequency of the approximate periodic solutions introduced by this condition are developed in this note. *M. Golomb.*

★**Popov, E. P. Improvement of the first approximation in studying self-oscillations of high order nonlinear systems.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 7 pp. $3.50.
Translation of the paper reviewed above

Popov, E. P. Auto-oscillations of nonlinear systems of high order with a slowly varying external force. Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 545–547. (Russian)
To determine the auto-oscillations of a system described by the equation $Q(p)x+R(p)F(x, px)=S(p)f(t)$, where $p=d/dt$, Q, R, S are polynomials with constant coefficients, F is a nonlinear function and $f(t)$ a slowly varying function, the author looks for approximate solutions of the form $x=a\sin(\int\omega dt)+x_0$ and uses his version of the method of harmonic linearization [see the paper reviewed fifth above] to determine a, ω and the component x_0 which corresponds to the slowly varying external force $f(t)$. *M. Golomb* (Lafayette, Ind.).

★**Popov, E. P. Self-oscillations of high order nonlinear systems with slowly varying external action.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 6 pp. (mimeographed) $3.00.
Translation of the paper reviewed above.

Popov, E. P. On small parameters in the method of harmonic linearization. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 2, 41–59. (Russian)
The author compares his version of the Krylov-Bogolyubov method of harmonic linearization for the approximate determination of periodic solutions in a regulating system containing nonlinear components [see the papers reviewed seventh and fourth above] with a version of Poincaré's method of small perturbations ascribed to Bulgakov and Letov [see, e.g., Letov, Prikl. Mat. Meh. **12** (1948), 149–156, 363–368; MR **9**, 588; **10**, 121]. By reasoning that is not convincing to this reviewer he comes to the conclusion that the two methods give in all cases results of the same degree of approximation, which are satisfactory even in cases where the nonlinearities are not small perturbations. There are also remarks to the effect that the presence of self-resonance and filtering in regulating systems justifies the approximations.
M. Golomb (Lafayette, Ind.).

Aĭzerman, M. A., and Gantmaher, F. R. On a class of dynamical problems reducing to the theory of relay systems. Prikl. Mat. Meh. **19** (1955), 222–224. (Russian)
The authors make the observation that the questions concerning periodic solutions and their stability for systems of the form

$$(1)\quad \sum_{j=1}^{n} d_{ij}(p)x_j - M_i f(x_k) = A_i \sin(\omega t+\varphi_i) \qquad (i=1, \cdots, n;\ p=d/dt),$$

where the $d_{ij}(p)$ are polynomials with constant coefficients and $f(x)=rx+a \operatorname{sign} x$ (M_i, A_i, φ_i, ω, r, a are given real numbers) can be reduced to the corresponding questions for "relay systems", where $r=0$, which have been studied more extensively. Obviously, the term $-rM_ix_k$ in (1) can be consolidated with the term $\sum d_{ij}(p)x_j$, thus changing (1) into a relay system. *M. Golomb* (Lafayette, Ind.).

★**Letov, A. M. Ustoĭčivost' nelineĭnyh reguliruemyh sistem. [Stability of nonlinear control systems.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 312 pp. 8.20 rubles.
The purpose of this book is to acquaint "a wide circle of specialists working in the field of automata with the possibilities of effective utilization of the direct method of Lyapunov in the solution of the foremost problems in the theory of automatic control." It is also pointed out that the book should be helpful to students and "aspirants" in the field of automatic control and applied mechanics. The book is confined to the presentation of results obtained during the last decade by Russian investigators on the problem of the "absolute" stability and the degree of stability of control systems. By "absolute" stability is meant stability where a function which characterizes the regulator is not determined precisely but is known to belong to some class of functions. The largest such class of functions considered is the class (A) described below. All methods are based upon the First and Second Theorem of Lyapunov. These theorems give sufficient conditions for stability and asymptotic stability, and their application depends upon the construction or discovery of suitable positive definite functions. The major part of the book is devoted to special methods for the construction of positive definite quadratic forms suitable for the class of control systems being studied and to the illustration of these methods on specific examples. The differential equations of the control systems are of the following general type:

$$\dot{y}_k=\sum_{i=1}^{n} b_{ki}y_k+n_ku \quad (k=1, \cdots, n),$$

$$V^2\ddot{u}+W\dot{u}+Su=r(s),$$

$$s=\sum_{i=1}^{n} p_iy_i-ru.$$

The first equation is for the system being controlled,

and the second equation is that of the regulator. The parameters b_{kl}, n_k, V, W, p_i, and r, are constants, though the non-stationary case is discussed briefly in Chapter XI. The function f may be nonlinear. The restrictions on f are: for some $s^*\geqq 0$, $f(s)=0$ for $|s|<s^*$ and $sf(s)>0$ for $|s|>s^*$, and f is continuous for $|s|\geqq s^*$. The class of all such functions is the class (A) referred to above. The Chapter headings are: Introduction. I) Equation of a control system. Statement of the problem of stability. II) First canonical form of the equations of a control system. III) Second and third canonical forms. IV) Stability of control systems. V) Construction of simplified stability criteria. VI) Strictly unstable control systems. VII) Programmed control. VIII) Problem of quality of regulation. IX) Stability of control systems with dual regulators. X) Two special problems in the theory of control systems. XI) Stability of non-stationary motions. The principal example used to illustrate the methods is Bulgakov's equation: $T^2\ddot{x}+U\dot{x}+kx+y=0$, $\dot{u}=f(s)$, and $s=ax+E\dot{x}+G^2\ddot{x}-u/l$.

The book is a plain, unsophisticated and painstakingly thorough treatise on the application of Lyapunov's direct method. The mathematics required of the reader is elementary. The author's notations are cumbersome but possess the advantage to the person wishing to apply the results that the formulas are self-explanatory. No matrix canonical form theorems are used, and the transformation of the basic differential equations to the various canonical forms and the computation of the transformations are discussed in great detail. Most of this is, of course, consistent with the purpose for which the book was published. *J. P. LaSalle* (Notre Dame, Ind.).

Križanovs'kiĭ, O. M. Determination of regions of values of the coefficients of a differential equation from bounds on the maximum frequencies of the natural vibrations in a control system. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 328–332. (Ukrainian. Russian summary)

Let a control system be described by linear differential equations with constant coefficients and assume the coefficients are to be determined so that there be no natural vibration of frequency higher than the given number $b\geqq 0$ ($b=0$ refers to the requirement that there be no oscillatory natural mode). This requirement is equivalent to the condition that all the roots of $H(z-ib)=0$, where $H(p)$ is the characteristic polynomial of the system, lie in the left half-plane. The author suggests that the well-known criteria for the latter occurrence be applied to the problem at hand. *M. Golomb*

Križanovs'kiĭ, O. M. Calculation of integral curves of optimality of control processes. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 183–190. (Ukrainian. Russian summary)

This paper deals with the evaluation of the integrals $I_k=\int_0^\infty (y^{(k)})^2dt$ ($k=0, 1, \cdots, n-1$), where $y=y(t)$ is the solution of the stable equation with constant coefficients $y^{(n)}+a_1y^{(n-1)}+\cdots+a_ny=0$ which satisfies the conditions $y^{(k)}(0)=y_k$. The I_k satisfy a simple algebraic system. The author establishes these systems in explicit form for the cases $n=1, 2, \cdots, 6$. *M. Golomb* (Lafayette, Ind.).

Križanovs'kiĭ, O. M. On choice of parameters of a control system which is described by a linear differential equation with constant coefficients. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 26–31. (Ukrainian. Russian summary)

Assume the constant parameters a_i of the differential operator $L(y)\equiv a_0y^{(n)}+\cdots+a_ny$ are to be chosen such that the transient $y(t)$ which satisfies $L(y)=0$ deviates, in the mean, as little as possible from a given function $Y(t)=\sum_{i=1}^{m}\sum_{j=0}^{k}A_{ij}t^je^{-\beta_it}$. This problem requires the calculation of the integral $J=\int_0^\infty [y(t)-Y(t)]^2dt$, where $y(t)$ is the solution of $L(y)=0$ for given $y^{(k)}(0)=y_k$, $k=0, \cdots, n-1$. The author evaluates J by the use of the Laplace transforms of $y(t)$ and $Y(t)$. *M. Golomb.*

Pliškin, Yu. M. On estimation of integral criteria of the quality of regulation of nonlinear systems. Avtomat. Telemeh. **16** (1955), 19–26. (Russian)

Let $(x_1(t), \cdots, x_n(t))$ be the transient in a nonlinear control system for which a Lyapunov function $V(t)$ is known. For a number of such systems the author finds an upper bound for the integral $\int_0^\infty [x_1^2(t)+\cdots+x_n^2(t)]dt$ in terms of $V(0)$. *M. Golomb* (Lafayette, Ind.).

Miroslavlev, E. N. Nonlinear systems with correcting structures. Vestnik Moskov. Univ. **9** (1954), no. 9, 33–40. (Russian)

The system considered is described by the following equations (simplified by the introduction of dimensionless quantities): $(D^3+D+k)\varphi+\eta=0$, $(r^2D^2+D)\eta=F(\sigma)$, $(T_2D+1)(\sigma+m\eta)=(T_1D+1)\varphi$. Here $D=d/dt$, and $F(\sigma)$, the nonlinear characteristic of the servomotor, is an odd bounded nondecreasing function. The author investigates the possible regimes if the system has functions of the feedback coefficient m and of the time constants T_1, T_2 of the correcting link. This is done by Poincaré's method of small parameters. The second of the above equations is replaced by $(r^2D^2+D)\eta-h\sigma=\mu[F(\sigma)-h(\sigma)]$, which is linear for $\mu=0$ and coincides with the original equation for $\mu=1$. Only first-order approximations are calculated. *M. Golomb* (Lafayette, Ind.).

Petrov, V. V., and Ulanov, G. M. On the theory of indirect regulation taking account of dry friction in a sensitive element. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 611–614. (Russian)

The control system discussed is described by the equations

$$T_a\dot{\varphi}=-\mu,$$
$$\delta\eta-\varphi=\varepsilon \operatorname{sign}\dot{\eta} \text{ when } \dot{\eta}\neq 0,$$
$$|\delta\eta-\varphi|<\varepsilon \text{ when } \dot{\eta}=0,$$
$$T_s\dot{\mu}=\operatorname{sign}(\eta-\gamma\mu),$$

where T_a, T_s, γ, δ, ε are constant parameters. The authors determine the possible regimes of this system by tracing the trajectories in the φ, $\dot{\varphi}$ surface which is made up of three superimposed sheets in the region $|\varphi+\gamma^{\delta T_a}\dot{\varphi}|<\varepsilon$, corresponding to the values 1, 0, -1 of sign $(\eta-\gamma\mu)$. *M. Golomb* (Lafayette, Ind.).

Pagni, Mauro. Equazioni differenziali lineari e problemi al contorno con condizioni integrali. Rend. Sem. Mat. Univ. Padova **24** (1955), 245–264.

The author presents two applications of the general alternative principle of G. Fichera [see Lezioni sulle trasformazioni lineari, vol. I, Ist. Mat., Univ., Trieste, 1954; MR **16**, 715]. The first application is the case of a boundary problem involving a linear system of first-order differential equations in $x_i(t)$ $(i=1, \cdots, n)$, a set of separated end-conditions at $x=a$, $x=b$, and m specified integral conditions. The second application involves a boundary problem determined by a linear partial differ-

ential equation of order n in a single function of r independent variables, a set of linear differential boundary conditions, and additional integral conditions. For the second application the basic assumption is that the alternative theorem holds for the boundary problem determined by the given differential equation with the differential boundary conditions, and the corresponding adjoint boundary problem, so that the contribution of the author is the derivation under these conditions of the corresponding alternative theorem for the more general system involving integral conditions.

It is to be remarked that for the problem of the first application one may formulate an equivalent boundary problem in functions $x_j(t)$ $(j=1, \cdots, n+m)$ and involving only two-point boundary conditions, and the usual alternative theorem for such boundary problems leads readily to the result given by the author.

W. T. Reid (Evanston, Ill.).

Hellman, O. Die Anwendung des Matrizanten bei Eigenwertaufgaben. Z. Angew. Math. Mech. **35** (1955), 300–315. (English, French and Russian summaries)

Let the solution of the vector-matrix system $dr/dt=A(t)r$ be written in the form $r(t)=\Omega_0^t(A)r(0)$. After outlining known properties of Ω, the author discusses the form of Ω when $A(t)$ has only one non-zero element in each row and column. He then gives matrix versions of standard arguments for self-adjoint differential equations of the second and fourth orders with boundary conditions; he estimates the first few eigen-values in some special cases.

F. V. Atkinson (Canberra).

Bihari, Imre. Generalisation of a theorem by Sturm and its application to Bessel functions and Legendre polynomials. Mat. Lapok **6** (1955), 165–175. (Hungarian. Russian and English summaries)

The author obtains Sturmian comparison theorems for the differential equations $y''+A_1(x)y'+B_1(x)y=0$, $Y''+A_2(x)Y'+B_2(x)Y=0$, where $A_2(x)\geqq A_1(x)$, $B_2(x)\geqq B_1(x)$. He uses his result to compare Bessel functions of different orders, or Legendre functions of different orders. In the case of Bessel functions he also obtains comparisons with elementary functions by taking one of the two orders involved to be half of an integer.

A. Erdélyi.

Gel'fand, I. M., and Levitan, B. M. On the determination of a differential equation from its spectral function. Amer. Math. Soc. Transl. (2) **1** (1955), 253–304.

Translated from Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 309–360; MR **13**, 558.

Kay, Irvin, and Moses, Harry E. The determination of the scattering potential from the spectral measure function. III. Calculation of the scattering potential from the scattering operator for the one-dimensional Schrödinger equation. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **CX-20** (1955), i+38 pp.

The problem treated by the first author [same report series, No. **EM-74** (1955); MR **16**, 1113] is here treated in the setting of the more abstract formulation of the authors [ibid., Nos. **CX-18** (1955); **CX-19** (1955); MR **16**, 929; **17**, 155]. "The advantage of the more general approach of the present paper is that it seems to indicate a way to tackle the more interesting problem of three-dimensional scattering from a general (i.e., non-symmetric potential."

N. Levinson (Cambridge, Mass.).

Laasonen, Pentti. Ein Problem bei der iterativen Bestimmung der Eigenwerte simultaner Differentialgleichungen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **195** (1955), 13 pp.

The definite and self-adjoint eigenvalue problem:

$$\frac{du(x)}{dx}=[F(x)+\lambda G(x)]\cdot u(x),$$

$$Au(a)+Bu(b)=0,\quad a\leqq x\leqq b,$$

with A, B, F, G as $n\times n$ matrices and with $u(x)$ as column-vector, is treated by the iteration

$$\frac{dV_i(x)}{dx}=F(x)V_i(x)+G(x)V_{i-1}(x),\quad AV_i(a)+BV_i(b)=0,$$

with V_k as $n\times m$ matrix. In order to find approximations to the first m eigenvalues, the matrices

$$Q_k=\int_a^b V'_i(x)S(x)V_{k-i}(x)dx$$

with $S(x)=T'(x)G(x)=G'(x)\cdot T(x)$ and with T as a suitable Bliss matrix are computed. The matrices Q_k represent sets of Schwarz constants. The matrix eigenvalue problem $\det(Q_{k-1}-\mu Q_k)$ furnishes m approximate eigenvalues $\mu_1, \mu_2, \cdots, \mu_m$ for the original problem. Remarks on convergence of these approximations when $k\to\infty$ conclude the paper. As a tool of investigation the author considers expansions of the matrices V_i in terms of the eigenelements of the original problem.

H. Bückner (Schenectady, N.Y.).

See also: Seidenberg, p. 454; Albrecht und Hochmuth, p. 467; Kato, p. 514; Pöschl, p. 539.

Partial Differential Equations

Hornich, Hans. Ueber nirgends lösbare lineare oder nichtlineare partielle Differentialgleichungen. Rend. Sem. Mat. Univ. Padova **24** (1955), 160–164.

Extending a previous result, the author shows that a function $\phi(x, y)$ can be constructed, in the entire plane, so that the equation:

$$(*)\qquad u_x+\phi(x, y)u_y=f(x, y)$$

has no solution in every region G in which f is continuous and f_y does not vanish identically. When f_y is identically 0, $f=f(x)$, and its integral is a trivial solution of (*); in particular, the homogeneous equation, when $f=0$ has only the trivial solution $u=$constant. The author also proves that the non-linear equation:

$$u_x+\phi(x, y)u_y=F(x, u, u),$$

where F, F_y, F_u are continuous for (x, y) in a region G and all u, has a solution U only if U_y vanishes identically in G; when this occurs, $U=U(x)$ has $F_y[x, y, U(x)]=0$ in G and $u=U(x)$ is a solution of the ordinary differential equation: $u'=F(x, y, u)$.

D. L. Bernstein.

Pucci, Carlo. Nuove ricerche sul problema di Cauchy. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **1** (1953), 45–67=Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **415** (1954), 23 pp.

This paper is concerned with solutions of problems of the form:

$$(1)\qquad u_{tt}(x, t)+a(x, t)u_{xx}(x, t)=f(x, t),$$

$$(2)\quad u(x, 0)=u_0(x,),\ u_t(x, 0)=u_i(x)\ (i=1, \cdots, m-1),$$

(u_{t^m} denotes $\partial^m u/\partial t^m$). If a function has derivatives of all orders with respect to x in a rectangle R, and if $|\phi_{x^s}(x, t)| \leqq M(s!)^\alpha/\varrho^s$, for $s=0, 1, 2, \cdots$, the function ϕ is said to be of class $G_\alpha[\alpha, \varrho, M]$. The author proves that under certain conditions on α and M, if the functions $a(x, t)$, $f(x, t)$, $u_i(x)$ are of class $G_\alpha[\alpha, \varrho, M]$, then there exists a unique solution of (1) and (2) in R which is of class $G_\alpha[\alpha, \varrho e^{-1-\alpha}, M]$. He also gives a formula for the solution as an infinite series whose terms are the results of applying repeatedly a certain linear operator to F, a function obtained from f and u_i by a specific formula. Other theorems in the paper are concerned with extending this result to a strip, with the relation between solution of two problems, where the coefficients in the equations of one dominate the corresponding coefficients in the other and like questions.

D. L. Bernstein (Rochester, N.Y.).

Saltykow, N. Intégrales de S. Lie des équations aux variables canoniques de seconde classe. Glas Srpske Akad. Nauka **206**. Od. Prirod.-Mat. Nauka (N.S.) **5** (1953), 71–78. (Serbo-Croatian. French summary)

If the equation $p_1+H(p_2, p_3, \cdots, p_n)=0$ ($p_k=\partial z/\partial x_k$) has a Lie integral of order k:

$$z=V(x_1, \cdots, x_{n-q}, C_1, \cdots, C_{n-1})+C_n,$$

$$x_{n-q+j}=\varphi_j(x_1, \cdots, x_{n-q}, C_1, \cdots, C_{n-1})$$

($j=1, 2, \cdots, q$; C_k=const.) it must be of the form $p_1+\sum_{j=1}^{q}\theta_j(p_2, \cdots, p_{n-q})p_{n-q+j}+\theta(p_2, \cdots, p_{n-q})$, where the θ_j, θ are arbitrary functions. The integral is then

$$z=-\theta(C_1, \cdots, C_{n-q-1})x_1+\sum_{k=1}^{n-q-1}C_k x_{k+1}+C_{n-q},$$

$$x_{n-q+j}=\theta_j(C_1, \cdots, C_{n-q-1})+C_{n-q+j}.$$

The method by which these results are obtained is described in many of the author's earlier publications [see, e.g., Methods of integration of partial equations of the first order with a single unknown, Serbian Acad. Sci., Belgrade, 1947; MR **10**, 253].

M. Golomb.

Goldhagen, E. Sur l'intégration explicite des équations aux derivées partielles du I^er ordre à deux fonctions inconnues de deux variables indépendantes. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 623–644. (Romanian. Russian and French summaries)

The author considers a partial differential equation of the first order for two unknown functions, z_1 and z_2, of two independent variables x and y, assuming that a general solution of this partial differential equation is of the form $z_i=f_i(x, y, \phi, \phi_x, \phi_y, \phi_{xx}, \cdots)$, $i=1, 2$, where f_1 and f_2 are given functions of x, y, of an "arbitrary" function $\phi(x, y)$, and a finite number of partial derivatives of ϕ. He shows that such an equation can be reduced to one of two canonical forms and determines the general solutions for these canonical forms.

A. Erdélyi.

★**Scott, W. T. A note on the hodograph transformation in three dimensions.** Non-linear differential equations of the second order, pp. 93–95. OOR Project No. 956, technical report. Northwestern University, Evanston, Ill., 1955.

The term hodograph transformation means an interchange of the roles of the independent and dependent variables. The author explores the possibility of using the hodograph transformation to linearize a system of three first-order quasilinear partial differential equations in three independent and three dependent variables. The result is the theorem: if given matrices C^i, D^{ij} are such that the matrices $(D^{ij})^T C^j$ ($i, j=1, 2, 3$) are skew symmetric, and if the matrices B^{ij} determined by

$$(D^{ij})^T C^j+[(D^{ij})^T C^j]^T=0 \quad (i, j=1, 2, 3),$$

$$B^{21}=(D^{12})^T C^2+[(D^{12})^T C^2]^T,$$

$$B^{12}=(D^{21})^T C^3+[(D^{12})^T C^1]^T,$$

$$B^{13}=(D^{12})^T C^1+[(D^{21})^T C^1]^T \quad (i=1, 2, 3)$$

are skew symmetric, then any solution of the linear system

$$C^1 x_1+C^2 x_2+C^3 x_3=0$$

with non-vanishing Jacobian is transformed by the hodograph transformation into a solution of the quasilinear system

$$(*) \qquad A^1 v_1+A^2 v_2+A^3 v_3=0$$

where the matrices A^i of (*) are determined from the matrices B^{ij} by

$$B^{ij}=\begin{pmatrix} 0, & -a^{3ij}, & a^{2ij} \\ a^{3ij}, & 0, & -a^{1ij} \\ -a^{2ij}, & a^{1ij}, & 0 \end{pmatrix} \quad (i, j=1, 2, 3).$$

M. Pinl (Cologne).

Yanenko, N. N. On discontinuities of solutions of quasilinear equations. Uspehi Mat. Nauk **10** (1955), no. 2(64), 195–202. (Russian)

Verfasser behandelt das quasilineare Differentialsystem

$$A_i(u, v, x, y)\frac{\partial u}{\partial x}+B_i(u, v, x, y)\frac{\partial u}{\partial y}$$

$$+C_i(u, v, x, y)\frac{\partial v}{\partial x}+D_i(u, v, x, y)\frac{\partial v}{\partial y}$$

$$+E_i(u, v, x, y)=0, \quad (i=1, 2)$$

unter folgenden Voraussetzungen: (1) $E_i(u, v, x, y)=0$ (homogenes System); (2) die Koeffizienten A_i, B_i, C_i, D_i hängen nur von u und v ab. Unter die Klasse solcher quasilinearer Differentialgleichungen fallen z.B. die hydrodynamischen Gleichungen

$$\varrho\frac{\partial v}{\partial t}+\varrho v\frac{\partial v}{\partial x}+f(\varrho)\frac{\partial \varrho}{\partial x}=0, \quad \varrho\frac{\partial v}{\partial x}+\frac{\partial \varrho}{\partial t}+v\frac{\partial \varrho}{\partial x}=0.$$

Für diese hatte bereits B. Riemann die Möglichkeit von Unstetigkeiten für die Lösungen $v(x, t)$, $\varrho(x, t)$ erkannt, auch wenn Anfangsbedingungen

$$\varrho(x, 0)=\varphi(x), \quad v(x, 0)=\psi(x), \quad -\infty<x<+\infty,$$

mit beliebig glatten Funktionen $\varphi(x)$ und $\psi(x)$ vorgeschrieben werden. Es besteht die Aufgabe, Bedingungen für die Ausbildung solcher Unstetigkeiten zu gewinnen. Mit Hilfe des Charakteristikensystems des zu Grunde liegenden quasilinearen Systems gewinnt Verfasser ein äquivalentes quasilineares System, das zugehörige „invariante System", der Form:

$$\frac{\partial r}{\partial y}+f_1(r, s)\frac{\partial r}{\partial x}=0, \quad \frac{\partial s}{\partial y}+f_2(r, s)\frac{\partial s}{\partial x}=0$$

mit den Anfangsbedingungen $r(x, 0)=\bar{\varphi}(x)$, $s(x, 0)=\psi(x)$. Die Funktionen $\bar{\varphi}(x)$ und $\bar{\psi}(x)$ sollen dabei in einem gewissen endlichen Bereich nicht verschwinden. Sind α und β charakteristische Parameter, so ergeben sich aus $dr/d\alpha=0$, $ds/d\beta=0$ die beiden Scharen der r- und s-Charakteristiken. Für $\partial f_1/\partial r>0$ verdichten sich nun die Charakteristiken im Abschnitt der vorderen Wellenfront und fließen ausein-

ander hinter der Wellenfront; im Falle $\partial f_1/\partial r<0$ liegen die Verhältnisse umgekehrt. Gilt:

$$f_1(r, s)=f_1(s),\ f_2(r, s)=f_2(r),$$

so verwandeln sich die Charakteristiken in parallele Geraden. Nur in diesem Falle treten keine Unstetigkeiten auf, woraus sich notwendige und hinreichende Bedingungen für das Auftreten von Unstetigkeiten ableiten lassen. Für die charakteristischen Parameter α, β und die abhängigen Veränderlichen x, y gewinnt Verfasser die Beziehungen

$$y=\alpha+\beta,\ x=h(\alpha)+k(\beta)$$

und damit auch die Lösung des Cauchyschen Problems des invarianten Systems. Zum Schluß untersucht Verfasser die Bedingungen für eine „Gradientenkatastrophe" (die Ableitungen $\partial r/\partial x$, $\cdots$, $\partial s/\partial y$ erreichen unbegrenzte Werte in der gleichen Zeit, in welcher r, s selbst in bestimmten Grenzen bleiben) und gewinnt notwendige Bedingungen für beschränkte Lösungen. Die Methode hat algebraischen Charakter und läßt sich aus diesem Grunde auf den Fall mehrerer unabhängiger Veränderlicher übertragen. *M. Pinl* (Köln).

★ **Webster, Arthur Gordon. Partial differential equations of mathematical physics.** Edited by Samuel J. Plimpton. 2d ed. Dover Publications, Inc., New York, 1955. vii+440 pp. $1.98.

Photo-offset reprint of the 2nd edition [Teubner, Leipzig, 1933].

★ **Diaz, J. B. Some recent results in linear partial differential equations.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 1–29. Edizioni Cremonese, Roma, 1955. 3000 Lire.

The purpose of the present article is to review some of the latest results obtained by a group of mathematicians at the Institute for Fluid Dynamics and Applied Mathematics at the University of Maryland. A generalization of Riemann's method for partial differential equations $L(u)=0$ has been given by J. B. Diaz and M. H. Martin. The starting point is the divergence identity

$$\left(\frac{v_x}{\Phi_x}-\frac{v_y}{\Phi_y}\right)L(u)+\left(\frac{u_x}{\Phi_x}-\frac{u_y}{\Phi_y}\right)M(v)=\left(\frac{u_xv_x}{\Phi_x}\right)_y-\left(\frac{u_yv_y}{\Phi_y}\right)_x,$$

where Φ is a solution of $L(u)=0$ for which $\Phi_x\neq 0$, $\Phi_y\neq 0$ and the associate operator

$$M(v)=v_{xy}-b\Phi_x^{-1}\Phi_yv_x-a\Phi_x\Phi_y^{-1}v_y.$$

J. B. Diaz and M. H. Martin obtained in this manner the known solution of the Cauchy problem for the wave equation

$$u_{x_1x_1}+\cdots+u_{x_nx_n}-u_{tt}=0\ (n \text{ even or odd}).$$

In several papers S. Bergman has developed a theory of integral operators for obtaining solutions of partial differential equations. As applied to

$$(*)\qquad L(u)\equiv u_{xx}+a(x, y)u_x+b(x, y)u_y+c(x, y)u=0,$$

his method is based on the following theorem: Let $E(x, y, t)$ be a solution of

$$(1-t^2)(E_{yt}+aE_t)-\frac{1}{t}(E_y+aE)+2xtL(E)=0,$$

such that, for $x\neq 0$,

$$\frac{(1-t^2)^{\frac{1}{2}}(E_y+aE)}{xt}$$

is continuous for $t=0$, and tends to zero for each (x, y) as t approaches ± 1. Then, if f is an arbitrary once continuously differentiable function, the function u defined by

$$u(x, y)=\int_{-1}^{+1}E(x, y, t, f)[\tfrac{1}{2}x(1-t^2)]\frac{dt}{\sqrt{(1-t^2)}}$$

is a solution of (*). The author discusses this theorem and some results of Le Roux. Le Roux's formulation is simpler in the fluid-dynamical application. Bergman's method consists in finding particular solutions of an associated equation for E, in three independent variables, whereas the method of Le Roux consists in determining particular solutions of the originally given differential equation in two independent variables. The theory of an approximation for the transonic flow of a polytropic gas has been developed by Diaz and Ludford. The transonic approximation is based upon first seeking all partial differential equations of the form

$$u_{yy}-yu_{xx}-f(y)u=0$$

which possess solutions of the following type:

$$u=A(y)B(\xi);\ \xi=-\xi\eta=\tfrac{4}{9}y^3-x^2,$$

where ξ and η are characteristic variables of Tricomi's equation $u_{yy}-yu_{xx}=0$. In a paper by Diaz and Ludford the following Cauchy problem has been solved:

$$L(u)=u_{xx}+u_{yy}-u_{tt}-\frac{k}{t}u_t-hu=0,\ k>0,\ t>0$$

$$u(x, y, 0)=g(x, y),\ u_t(x, y, 0)=0.$$

For k any real number this singular Cauchy problem was first solved by A. Weinstein by his "method of recurrence" and a generalized method of descent. Schwarz's inequality can be used to obtain upper and lower bounds for the Dirichlet integral of a solution of Neumann's problem

$$\int_D(v_x^2+v_y^2)dx\,dy,\ \Delta v=0\ \text{ on } D,\ \frac{\partial v}{\partial n}=f \text{ on } C,$$

where D is a bounded, plane, connected, open set with a smooth boundary C and f is a given function on C. A systematic procedure for obtaining monotone sequences of upper and lower bounds for quadratic integrals is given by Diaz. A somewhat different order of ideas has been exploited by L. E. Payne and H. F. Weinberger for the determination of upper and lower bounds for Dirichlet integrals, harmonic and biharmonic functions.

M. Pinl (Cologne).

Vilcovici, V. Sur le théorème des valeurs extrêmes (TVE). Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 741–749. (Romanian. Russian and French summaries)

L'Auteur étend ce théorème bien connu, concernant l'équation de la chaleur $u_{xx}=u_t$, aux solutions des équations ayant les formes générales

$$\Delta u=\phi(t, x_j, u, u_{x_k}, u_t) \text{ et } u_t=\psi(t, x_j, u, u_{x_k}, \Delta u)$$

($\Delta u=\sum_{i=1}^{n}u_{x_ix_i}$), dans un espace à $n+1$ dimensions et pour un domaine D limité, quelconque, en montrant que la solution d'une pareille équation prend sur le contour ses valeurs extrêmes. On en déduit ensuite un théorème d'unicité concernant la solution d'une équation ayant la

forme générale

$$u_t=a(t, x_j)\Delta u+\sum_{i=1}^{n} b_i(t, x_j)u_{x_i}+c(t, x_j)u+g(t, x_j),$$

si la solution prend des values données sur la frontière.

Résumé de l'auteur.

★ **Eidus, D. M. On a boundary problem for $\Delta u+\lambda^2 u=0$.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 4 pp. $2.00.

Translated from Doklady Akad. Nauk SSSR (N.S.) **100** (1955), 631–633; MR **17**, 268.

Goertzel, Herbert. On the fundamental eigenvalue of a solid. Oak Ridge National Laboratory, Oak Ridge, Tenn., Rep. **ORNL 1892** (1955), 73 pp.

This report contains three disjoint results concerning the lowest eigenvalue λ_1 of the membrane problem $\Delta u+\lambda u=0$ on an n-dimensional region with $u=0$ on the boundary. It is first proved by Steiner symmetrization [G. Pólya and G. Szegö, Isoperimetric inequalities in mathematical physics, Princeton, 1951, p. 156; MR **13**, 270] that of all n-dimensional regions of given n-volume the hypersphere has the smallest value of λ_1. This result has been proved in a more elegant manner (by Schwarz symmetrization) by E. Krahn [Acta Comment. Univ. Tartuensis. A. **9** (1926), no. 1]. Secondly, the lowest eigenvalues for some (two-dimensional) circular segments are approximated by a routine finite-difference procedure with no error estimate. Finally, the three-dimensional membrane problem with interface is formulated in a way that is incorrect except in those trivial cases where the problem is essentially one-dimensional (concentric spheres or parallel infinite slabs). Two such cases, which reduce to equations involving trigonometric functions, are treated numerically.

H. F. Weinberger (College Park, Md.).

Boboc, N. Sur un théorème de type Sturm et applications au problème de la séparation des zéros des fonctions propres de l'opérateur Δ. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 723–729. (Romanian. Russian and French summaries)

Consider the membrane eigenvalue problem: $u_{xx}+u_{yy}+A(x, y, \lambda)u=0$ in a plane domain Ω, subject to the boundary condition $u=0$ on the boundary Γ, where $A(x, y, \lambda)>0$ and increases monotonically with λ. Let $\lambda_1\leqq\lambda_2\leqq\cdots\leqq\lambda_m\leqq\cdots$ denote the eigenvalues and $\varphi_1, \varphi_2, \cdots, \varphi_m, \cdots$ the corresponding eigenfunctions. I. G. Petrovskiĭ [Lectures on partial differential equations, Gostehizdat, Moscow, 1950; MR **13**, 241] has proposed the problem of deriving theorems of Sturm's type in this situation. The author introduces a concept of "separation (C)" and proves the following two theorems: Theorem 1: If $\lambda_s<\lambda_k$, then the zeros of φ_k "separate (C)" the zeros of φ_s. Theorem 2: The zeros of two linearly independent eigenfunctions corresponding to the same (multiple) eigenvalue "separate each other (C)".

J. B. Diaz (College Park, Md.).

Duff, G. F. D. A quasi-linear boundary value problem. Trans. Roy. Soc. Canada. Sect. III. (3) **49** (1955), 7–17.

This paper proves existence and uniqueness for the function u defined by the quasi-linear elliptic partial differential equation

$$\Delta u=F(P, u), \tag{1}$$

where P is a point in the n-dimensional space of the independent variables x^i, $\Delta u\equiv a^{-\frac{1}{2}}(\partial/\partial x^i)[a^{\frac{1}{2}}a^{ik}(\partial/\partial x^k)u]$ and a^{ik} is a suitable contravariant tensor field, with quasi-linear boundary condition

$$\frac{\partial u}{\partial n}+B(P, u)=f(P). \tag{2}$$

Equation (1) holds for a compact domain in N-space, while equation (2) holds on the boundary. In order to obtain an a priori bound on u, it is necessary to require that

$$\frac{\partial F}{\partial u}\geqq\delta>0, \quad \frac{\partial B}{\partial u}>0.$$

This hypothesis on B_u permits one to establish uniqueness by applying Green's first formula to the region (if any) where $w\equiv u_2-u_1>0$.

The proof of existence is obtained by a use of the "first variation" or "method of perturbations." The perturbation satisfies, as usual, linear equations, and one seeks to "overlap" these local linearizations and take a suitable limit. This is accomplished by: (i) replacing the datum function $f(P)$ by $tf(P)$, so that, for $t=0$, one has the (unique) solution $u=0$, while $t=1$ gives the original problem; (ii) observing that $v(P, t)\equiv\partial u(P, t)/\partial t$ is now determined by $v(P, t)=\int R(P, Q, t)f(Q)dS_Q$, where the Robin's function $R(P, Q, t)$ depends upon $u(P, t)$; (iii) in this way obtaining a system of two simultaneous integro-differential equations; (iv) solving these by successive approximations; (v) proving convergence by means of an a priori bound on u. The paper extends a method of Bergman and Schiffer. *R. B. Davis* (Durham, N.H.).

Kreyszig, Erwin. On a class of partial differential equations. J. Rational Mech. Anal. **4** (1955), 907–923.

Bergman's integral operators possess the property of mapping some class of analytic functions into the set of solutions of some partial differential equation, thereby permitting function-theoretic notions to be used in the study of solutions. Integral operators of various types exist, either mapping analytic functions of a single complex variable into solutions of a second-order equation in two independent variables, or else mapping functions of two complex variables into solutions of a differential equation with three independent variables [S. Bergman, same J. **3** (1954), 539–560; MR **16**, 705]. Integral operators are subdivided further in various ways. The present paper considers an equation in two independent variables:

$$\Delta\psi+\alpha(x, y)\psi_x+\beta(x, y)\psi_y+\gamma(x, y)\psi=0 \tag{1}$$

which can be re-written:

$$u_{zz^*}+B(x, z^*)u_{z^*}+C(z, z^*)u=0. \tag{2}$$

A special class of integral operators is considered, known as operators of exponential type, defined by

$$u(z, z^*)=\int_{-1}^{1} E(z, z^*, t)f\{z(1-t^2)/2\}(1-t^2)^{-\frac{1}{2}}dt,$$

where $u(z, z^*)$ is a solution of (2), and f is the "associated analytic function" which is mapped into u, and where the known fixed kernel $E(z, z^*, t)$ is given by

$$E=e^Q, \quad Q(z, z^*, t)=\sum_{\mu=0}^{n} q_\mu(z, z^*)t^\mu.$$

The main theorem of the paper presents necessary and sufficient conditions on the coefficients B and C in order that the solutions $u(z, z^*)$ of equation (2) may be gener-

ated by a Bergman operator of exponential type. The proof of the theorem is achieved by substitution, leading to an involved system of $(2m+3)$ linear partial differential equations, which is then solved by an ingenious step-wise procedure. An example of Bergman is cited to illustrate the use of integral operators in the study of special functions. *R. B. Davis* (Durham, N.H.).

Casci, Corrado. Ancora sulla distribuzione della temperatura in un anello rotante in ambienti a temperatura diversa. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 368–372.

A ring is placed in two surrounding media of different but constant temperatures. The problem of finding the temperature of the ring when stationary or when rotating with constant velocity was treated earlier by the author [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Nat. (8) 7 (1950), 219–224, 297–303; MR **11**, 522]. A simpler and more intuitive approach to these same problems is set forth in this paper. *F. G. Dressel* (Durham, N.C.).

Jörgens, Konrad. Harmonische Abbildungen und die Differentialgleichung $rt-s^2=1$. Math. Ann. **129** (1955) 330–344.

The author establishes the theorem: Let $\{x=\mathrm{Re}\, F_1(w), y=\mathrm{Re}\, F_2(w)\}$, $w=u+iv$, define a 1–1 harmonic mapping of the annulus $\Delta: 1<|w|<\lambda$ on a region D of the (x, y)-plane which contains the annulus $R_1^2\leqq x^2+y^2\leqq R_2^2$ such that at least one boundary point of D is interior to the disc $x^2+y^2<R_1^2$. Let Γ_1 denote the image in Δ of the circle $x^2+y^2=R_1^2$. Then $R_2^2/12<R_1^2+J^2(1+\log^2\lambda+\lambda^2)$, where $J=(2\pi)^{-1}\oint_{\Gamma_1}\{|dF_1(w)|+|dF_2(w)|\}$. The proof is based on an inequality of E. Heinz on the Fourier coefficients of a monotone function.

As a consequence of his theorem, the author obtains new proofs of results of Bers [Ann. of Math. (2) **53** (1951), 364–386, esp. lemmas 3.6 and 3.7; MR **13**, 244] on harmonic mappings of doubly connected regions. These results are used to discuss isolated singularities of single-valued solutions of (*) $z_{xx}z_{yy}-z_{xy}^2=1$. Since for any solution $z(x, y)$ of (*), z_x and z_y are monotone in x and y, respectively, these functions are evidently bounded at an isolated singular point. The author proves that if, in addition, z_x (or z_y) is continuous at such a point, then the singularity is removable. He also shows that if $z(x, y)$ is a solution of (*) for all (x, y) with $x^2+y^2>0$, then $z(x, y)$ can be obtained by a simple transformation from the solution

$$z'(x, y)=\int_0^{\sqrt{(x^2+y^2)}}\sqrt{(1+\sigma^2)}\,d\sigma.$$

R. Finn (Los Angeles, Calif.).

Aleksandrov, A. D. Some theorems on partial differential equations of the second order. Vestnik Leningrad. Univ. 9 (1954), no. 8, 3–17. (Russian)

Let $F(x, z)=F(x_1, \cdots, x_n, z, z_1, \cdots, z_n, z_{11}, z_{12}, \cdots, z_{nn})$ be continuous in all variables and of class C' in z, z_i, z_{ik}. We say that $F(x, z)$ is, for a given function $u(x)$ of class C'' (defined in a domain G of x-space), elliptic at a point $x_0\in G$ if, with $u_i=\partial u/\partial x_i$, $u_{ik}=\partial^2 u/\partial x_i x_k$ and $F_{ik}=\partial^2 F/\partial z_i\partial z_k$, the form $\sum F_{ik}\xi_i\xi_k$ is positive definite for $x=x_0$. (Observe that usually negative definiteness is also admitted for the elliptic character.) The main result is the following: If $u(x)$ and $v(x)$ are two solutions of $F(x, z)=0$ defined in G and $F(x, z)$ is elliptic at $x_0\in G$ for $z=(1-t)u+tv$ and all $0\leqq t\leqq 1$, then $u(x)-v(x)$ does not attain in the interior of G an absolute minimum or maximum different from 0. This theorem was used by the author and Sen'kin for rigidity problems on convex surfaces [same Vestnik **10** (1955), no. 8, 3–13; MR **17**, 295] and it contains as special case the theorem of E. Hopf that a linear homogeneous elliptic equation cannot have a solution different from zero whose minimum or maximum is zero. *H. Busemann* (Los Angeles, Calif.).

Mihlin, S. G. Degenerate elliptic equations. Vestnik Leningrad. Univ. 9 (1954), no. 8, 19–48. (Russian)

In m-dimensional real Euclidean space consider a bounded open set Ω, and let $u(x)=u(x_1, \cdots, x_m)$ be a real-valued function defined on Ω and having generalized second order partial derivatives [in the sense of S. L. Sobolev, Some applications of functional analysis, Leningrad. Gos. Univ., 1950; MR **14**, 565] in Ω. Consider the differential operator

$$Lu=-\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{m}B_i\frac{\partial u}{\partial x_i}+Cu,$$

where the coefficients are real-valued and bounded on the closure of Ω. The operator L is supposed to be elliptic in Ω (i.e., Au the eigenvalues of the matrix $\|A_{ij}(x)\|$ are positive, for any x in Ω) and degenerately elliptic on the boundary of Ω (i.e. for any x on the boundary all the eigenvalues of $\|A_{ij}(x)\|$ are non-negative, and for at least one such x one has zero as an eigenvalue). The "degenerate" subset of the boundary (the subset where at least one eigenvalue is zero) is supposed to be contained in an $(m-1)$-dimensional hyperplane. The present paper is devoted to the study of the spectrum of the operator L, under suitable boundary conditions. The main results were formulated earlier in a less general form in two notes [Dokl. Akad. Nauk SSSR (N.S.), **91** (1953), 723–726; **94** (1954), 183–185; MR **16**, 366, 367]. Several of the results, relative to the case of a discrete spectrum, were also obtained independently by M. I. Višik [ibid. **93** (1953), 9–12, 225–228; MR **15**, 798]. *J. B. Diaz.*

★ **Miranda, C. Systèmes elliptiques d'équations linéaires aux dérivées partielles du premier ordre.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 30–38. Edizioni Cremonese, Roma, 1955. 3000 Lire.

The system of two first-order linear equations

$$(1)\quad a_{i1}u_x+a_{i2}u_y+b_{i1}v_x+b_{i2}v_y=c_{i1}u+c_{i2}v+f_i\quad(i=1, 2),$$

where the a_{ij}, b_{ij}, c_{ij}, and f_i are functions of x and y, is called elliptic if the quadratic form

$$\sum_{i,k=1}^{2}\begin{vmatrix}a_{1i} & b_{1k}\\ a_{2i} & b_{2k}\end{vmatrix}\lambda_i\lambda_k$$

is positive definite. The present paper is a brief but clear exposition of such elliptic systems in relation to boundary-value problems. As such it omits, by deliberate intent, consideration of the system (1) from the point of view of the Cauchy problem (as in the work of Carleman and Douglis) or the generalized theory of functions (Bers, Gelbart, Petrovsky, Douglis).

By a theorem of Hellwig [Math. Z. **56** (1952), 388–408; MR **14**, 985], the system (1) can be reduced to the cannonical form

$$u_x-v_y=c_{11}u+c_{12}v+f_1,$$
$$u_y+v_x=c_{21}u+c_{22}v+f_2.$$

Among further results, the relation of elliptic systems to

problems in the calculus of variations is considered. Emphasis is placed on a paper of Miranda [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) 3 (1952), 85–121; MR 14, 175] and on certain results of Morrey, Hellwig, and Haack. *R. B. Davis.*

Yosida, Kôsaku. A characterization of the second order elliptic differential operators. Proc. Japan Acad. 31 (1955), 406–409.

The purpose of this note is to characterize second-order elliptic operators in an m-dimensional C^∞ Riemann space R essentially by two basic properties, namely: (i) such an operator E is local in that $Ef(x_0)$ depends only on the values of $f(x)$ in a neighborhood of x_0; and (ii) if $f(x)$ has a local minimum at x_0 then $Ef(x_0)\geqq 0$. For operators satisfying (i) and (ii) plus (iii) $E1=0$, W. Feller [Ann. of Math. (2) 60 (1954), 417–436; 61 (1955), 90–105; MR 16, 488, 824] has shown in the one-dimensional case that E can be represented by repeated differentiation: $Ef=D_vD_uf$, with respect to monotone non-decreasing functions u and v. In generalizing Feller's result to m-dimensions, Yosida chooses to circumvent this generalized notion of differentiation by imposing certain regularity conditions on the domain of E. In order to describe these conditions, let G denote the Lie group of isometries on R, assumed transitive. Further let $D(E; x_0)$ denote the set of functions which behave locally about x_0 like a function in the domain of E and define Ef accordingly for such functions. Let $C^\infty(x_0)$ be the set of functions which behave locally near x_0 like a function in C^∞. With these conventions, the author assumes (iv) that $D(E; x_0)\cap C^\infty(x_0)$ is dense in $C^0(x_0)$, (v) that E is "locally" closed at x_0, and (vi) for $f\in D(E; x_0)$ that $f_\alpha(x)=f(T_\alpha x)\in D(E; x_0)$ for $T_\alpha\in G$ in some neighborhood of the identity T_{α_0} and that $Ef_\alpha(x)$ is continuous in (x, α) in some neighborhood of (x_0, α_0). He then proves that there exist numbers $b^i(x_0)$, $a^{ij}(x_0)$ such that for $f\in D(E; x_0)\cap C^\infty(x_0)$, Ef has the representation

$$Ef(x_0)=a^{ij}(x_0)\frac{\partial^2 f}{\partial x_0{}^i\partial x_0{}^j}+b^i(x_0)\frac{\partial f}{\partial x_0{}^i}$$

and $(a^{ij}(x_0))$ is a positive hermitean matrix.

R. S. Phillips (Los Angeles, Calif.).

Lopatinskiĭ, Ya. B. On a method of reducing boundary problems for a system of differential equations of elliptic type to regular integral equations. Ukrain. Mat. Ž. 5 (1953), 123–151. (Russian)

Definitionen: $u=\begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix}$;

$$A(x,\alpha)=\sum_{j_1+\cdots+j_n=0}^{s_{kl}} a_{j_1\cdots j_n}^{kl}(x)\alpha_1^{j_1}\cdots\alpha_n^{j_n}\ (k,l=1,\cdots,p);$$

$B(x, \partial/\partial x)$ ist eine $\frac{1}{2}ps\times p$-Operatorenmatrix; $s=\max_{k,l} s_{kl}$. Die Elemente der j-ten Zeile, $B_j(x,\beta)$, sind β-formen des Grades s_j $(j=1, 2, \cdots, \frac{1}{2}ps)$, $s_j<s$; $\underline{s}\,(\bar{s})=\min_j(\max_j)\, s_j$; E ist die $p\times p$-Einheitsmatrix. $A_0(x, \partial/\partial x)$, $B_0(x, \partial/\partial x)$ Matrizen der höchsten Ordnung in $A(x, \partial/\partial x)$, bzw. $B(x, \partial/\partial x)$, K^* ein geschlossener Weg in der komplexen λ-Halbebene, der alle Wurzeln der Gleichung: $\det(A_0(y,\tau+\lambda\nu(y)))=0$ umschließt. V der Beschränkte konvexe Bereich des n-dimensionalen Raumes mit dem Liapunoffschen Rande ∂V. Es wird der folgende Hauptsatz bewiesen: Voraussetzungen: 1) ∂V ist $(\bar{s}-\underline{s})$-mal stetig differenzierbar; 2) die Koeffizienten bei den Ableitungen der j-ten Ordnung in $A(x, \partial/\partial x)$ sind $j+(\bar{s}-\underline{s})$-mal stetig differenzierbar $(j=0, \cdots, s)$; 3) die Koeffizienten der j-ten Ordnung in B_k gehören zu $C^{j+(\bar{s}-\underline{s}-s_k)}$;

4) $\operatorname{Rang}\int_{K^*} B_0(y, \tau+\lambda\nu(y))A_0^{-1}(y, \tau+\lambda\nu(y))\times(E, \cdots, \lambda^{s-1}E)\partial\lambda=\frac{1}{2}ps$;

$\nu(y)$ der normale Einheitsvektor in $y\in\partial V$; $\tau(\nu(y))=0$; $|\tau|\neq 0$; 5) $f_k(y)\in C^{\bar{s}-s_k}(\partial V)$. Behauptung: Die Randwertaufgabe:

$$A\left(x,\frac{\partial}{\partial x}\right)u(x)=0,\ \lim_{x\to y\in\partial V} B\left(y,\frac{\partial}{\partial x}\right)u(x)=f(y)$$

ist zurückführbar auf ein System von regulären Integralgleichungen (ähnliches Ergebnis für den Halbraum). Methode: eine Verfeinerung der Methode von Levi-Sternberg, wobei die in einer früheren Abhandlung des Verf. konstruierte Fundamentalmatrix herangezogen wird [Ukrain. Mat. Ž. 3 (1951), 3–38, 290–316; MR 16, 928, 256]. Anmerkung des Ref.: Da der Nachweis der Existenz der Fundamentalmatrix im Großen nicht erbracht wurde [vgl. Gihman, ibid. 3 (1951), 317–339, S. 338; MR 14, 1101], muß man in dem „Hauptsatz" noch die folgende Voraussetzung hinzufügen: 6) V ist der Bereich, in dem die Fundamentalmatrix existiert.

K. Maurin (Zbl 52, 102).

Lopatins'kiĭ, Ya. B. On a method of reducing boundary problems for a system of differential equations of elliptic type to regular integral equations. Dopovidi Akad. Nauk Ukrain. RSR 1952, 381–388. (Ukrainian. Russian summary)

See the paper reviewed above.

Cronin, Jane. The Dirichlet problem for nonlinear elliptic equations. Pacific J. Math. 5 (1955), 335–344.

The problem of inverting a nonlinear analytic transformation in real and complex Euclidean space is quite different. In particular, in the complex case the inversion is always possible. In this paper the author shows that the state of affairs is the same for transformations of a complex Banach space of the form $I+C+T$, where C is linear and completely continuous, T nonlinear, analytic and of higher order. This result is applied to the theory of nonlinear elliptic equations. Main theorem: Let F be an analytic function, $z=z_0$ a solution of the equation $F(x, y, z, p, q, r, s, t)=\psi$ with $\psi=\psi_0$, and assume that the equation is elliptic at $z=z_0$. Conclusion: The first boundary-value problem for the above equation will have a complex solution if the boundary data are close enough to those of z_0, and ψ to ψ_0. *P. D. Lax.*

Stellmacher, Karl L. Eine Klasse huyghenscher Differentialgleichungen und ihre Integration. Math. Ann. 130 (1955), 219–233.

A linear partial differential equation of the second order may be said to be of Huygens' type if Huygens' Principle holds for it in the strict sense. By this is meant that the solution of the Cauchy initial-value problem at some point P depends only on the initial data on the intersection of the characteristic conoid with vertex P and the space-like surface which carries the data.

Hadamard [Lectures on Cauchy's problem, Yale Univ. Press, 1923] proved that if an equation is of Huygens' type, the number n of independent variables must be even and the elementary solution must contain no

logarithmic term. Mathisson [Acta Math. **71** (1939), 249–282; MR **1**, 120] showed that, when $n=4$, an equation of Huygens' type must be reducible to the wave equation by suitable changes of the variables.

In the present paper, equations of the form

$$\frac{\partial^2 u}{\partial t^2}+\frac{\lambda_0}{t^2}u-\sum_{i=1}^{m}\left(\frac{\partial^2 u}{\partial x_i^2}+\frac{\lambda_i}{x_i^2}u\right)=0$$

are considered when (1) $m+1$ is an even number $\geqq 6$, (2) $\lambda_i=-\nu_i(\nu_i+1)$, where ν_i is a non-negative integer. It is shown that such an equation is of Huygens' type if and only if (3) $\sum_{i=0}^{m}\nu_i\leq\frac{1}{2}(m-3)$.

Condition (2) is introduced so that Asgeirsson's Mean-Value Theorem may be used to get the solution of the equation. Then the elementary solution is constructed to justify condition (2), the author shows for no other forms of λ_i can the equation be of Huygens' type.

E. T. Copson (St. Andrews).

★**Andreev, N. N. Neutralization method in solving wave problems.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 10 pp. $6.00.

Translated from Collection in honor of the 70th birthday of A. F. Ioffe, pp. 467–473 [Akad. Nauk SSSR, Moscow, 1950; MR **16**, 885].

★**Lahaye, Edmond. Sur la résolution d'une équation linéaire hyperbolique du second ordre par l'introduction d'une variable paramétrique.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 10–13. Fédération belge des Sociétés Scientifiques, Bruxelles.

The author solves the Cauchy problem for the wave equation by imbedding the wave operator in a one-parameter family of operators $L_\tau u=0$, τ small, by integrating the transformation $L_0 u=(L_\tau-L_0)v$; the method converges if (and, it seems to the reviewer, only if) the data are analytic.

P. Lax (New York, N.Y.).

Ladyženskaya, O. A. On solvability of the fundamental boundary problems for equations of parabolic and hyperbolic type. Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 395–398. (Russian)

Let Ω be a domain in R^n with smooth boundary Γ and let Q be a cylinder in $R^n\times R$ with base Ω. The operator S in $L^2(Q)$ defined by $Su=\partial u/\partial t-\Delta u$ for all $u(x,t)$, such that $\partial u/\partial t$, $\partial u/\partial x^i$, $\partial^2 u/\partial x^i\partial x^k$ (defined in the weak sense) are in $L^2(Q)$ and $u(x,0)=0$; $u(x,t)=0$ for $x\in\Gamma$, corresponds to a mixed boundary value problem for the equation of heat. Using the estimates of a note by the author [same Dokl. (N.S.) **79** (1951), 723–725; MR **14**, 280] and a uniqueness proof for weak solutions of the mixed problem it is proved that S is closed and that S^{-1} is bounded and defined everywhere. — Let $Tu=\partial^2 u/\partial t^2-\Delta u$ for all u with second derivatives in L^2 and $u(x,t)=0$, $x\in\Gamma$; $u(x,0)=\partial u(x,0)/\partial t=0$ the closure of T has bounded inverse defined everywhere in L^2. (This result is not new.) Analogous results are stated but not proved in detail when the Laplacian is replaced by a general elliptic second order differential operator.

L. Hörmander (Zbl **56**, 95).

Mizohata, Sigeru. Ellipticité des équations paraboliques. C. R. Acad. Sci. Paris **241** (1955), 1547–1549.

Un système d'équations aux dérivées partielles

$$\frac{\partial^{n_i}u_i}{\partial t^{n_i}}=\sum_{j=1}^{N}\sum_{(k)}a_{ij}^{k_0k_1,\cdots,k_n}(x,t)\frac{\partial^{k_0+k_1+\cdots+k_n}}{\partial t^{k_0}\partial x_1^{k_1}\cdots\partial x_n^{k_n}}u_j+g_i(x,t),$$

$$i=1,2,\cdots,N,\ k_0<n_j,\ n_i>0,$$

parabolique au sens de Petrowsky [Bull. Univ. d'Etat Moscou. Ser. Internat. Sect. A. **1** (1938), no. 7] dans un ouvert Ω donné est elliptique si les coefficients a et les fonctions g sont indéfiniment dérivables dans Ω; autrement dit, tout vecteur-distribution u solution du système étudié a nécessairement pour composantes des fonctions u_i indéfiniment dérivables dans Ω. Les démonstrations sont esquissées pour $N=1$, $n_i=1$. *H. G. Garnir* (Liège).

Conte, S. D. On some non-linear partial differential equations. Indust. Math. **4** (1953), 17–21.

The author solves the initial-value problem for non-linear parabolic equations of the form $u_t=ku_{xx}+P(u)u_x^2$ by introducing a new unknown as nonlinear function of the old, in terms of which the equation becomes the heat equation. In case $P(u)=$constant, this procedure has been carried out by E. Hopf [Comm. Pure. Appl. Math. **3** (1950), 201–230; MR **13**, 846] and by J. Cole [Quart. Appl. Math. **9** (1951), 225–236; MR **13**, 178]; in this case the equation becomes, after differentiation with respect to x and introduction of $v=u_x$ as new unknown, Burgers' model for turbulence [Advances in Appl. Mech., v. 1, Academic Press, New York, 1948, pp. 171–199; MR **10**, 270].

P. D. Lax (New York, N.Y.).

Smirnov, M. M. On a boundary problem for an equation of mixed type. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 699–701. (Russian)

Dans le plan (x,y), soit $A=(0,0)$, $B=(1,0)$, $C=(\frac{1}{2},-\frac{1}{2})$; soit D un ouvert limité par AC, CB, et un arc de Jordan σ joignant A et B dans $y>0$. On cherche y dans D solution de

$$\frac{\partial^4}{\partial x^4}u+2\operatorname{sgn} y\frac{\partial^4}{\partial x^2\partial y^2}u+\frac{\partial^4}{\partial y^4}u=0,$$

avec les données de Dirichlet sur σ (u et $\partial u/\partial n$ donnés sur σ) et avec $\partial u/\partial n$ donné sur AC et BC (l'A. précise les conditions de régularité des données et requises pour u). Dans ces conditions u existe et est unique. Démonstration: on résout d'abord le problème dans $y<0$, où la solution fait intervenir deux fonctions arbitraires; les combinaisons

$$u_{xx}-u_{yy}|_{y=0}\text{ et }\frac{\partial}{\partial y}(u_{xx}-u_{yy})|_{y=0}$$

éliminent ces deux fonctions; on est alors ramené à chercher u dans $y>0$ avec les données de Dirichlet sur σ, $u_{xx}-u_{yy}$ et $\partial(u_{xx}-u_{yy})/\partial y$ sur AB; ce dernier problème admet une solution unique (l'A. utilise à cet effect la méthode de Vekua-Mushelišvili [cf. N. P. Vekua, Systèmes d'équations intégrales singulières, Gostehizdat, Moscou-Leningrad, 1950; MR **13**, 247]); on détermine alors u pour $y<0$ par les données de Cauchy sur $y=0$, d'où le résultat.

J. L. Lions (Nancy).

Šilov, G. E. On conditions of correctness of Cauchy's problem for systems of partial differential equations with constant coefficients. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 89–100. (Russian)

Suite des travaux de Tihonov [Mat. Sb. **42** (1935), 199–216], Petrowsky [Bull. Univ. d'Etat Moscou. Ser. Internat. Sect. A. **1** (1938), no. 7], Gårding [Acta Math. **85** (1951), 1–62; MR **12**, 831], Schwartz [Ann. Inst. Fourier, Grenoble **2** (1950), 19–49; MR **13**, 242], Gel'fand et Šilov [Uspehi Mat. Nauk (N.S.) **8** (1953), no. 6(58), 3–54; MR **15**, 867], et Kostyučenko et Šilov [ibid. **9** (1954), no. 3(61), 141–148; MR **16**, 253], cet article est

probablement antérieur à Gel'fand et Šilov, Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 1065–1068 [MR **17**, 267] et Kostyučenko, ibid. **103** (1955), 13–16 [MR **17**, 267].

Les démonstrations complète seront données ailleurs. On cherche $u_j(x, t)$ $(j=1, \cdots, m)$, solution dans $t \geqq 0$ de

$$(1) \qquad \frac{\partial u_j(x,t)}{\partial t} = \sum_k P_{jk}\Big(\frac{1}{i}\frac{\partial}{\partial x}\Big)u_k(x, t),\ x \in R,$$

où P_{jk}=polynome à coefficients constants de degré $\leqq p$, avec (2) $u_j(x, 0)=U_j(x)$ donnée (tout vaut avec $x \in R^n$, et des coefficients dépendants de t). Soit $Q(s, t)$ la matrice fondamentale du système différentiel associé à (1) par transformation de Fourier en x; la matrice de Green $G(x, t)$ du problème de Cauchy (1), (2), s'obtient formellement par transformation de Fourier inverse. Soit $\lambda_k(s)$ les valeurs propres de $\|P_{jk}(s)\|$; posons $\Lambda(s)=\max_k \operatorname{Re} \lambda_k(s)$. On a:

$$\exp t\Lambda(s) \leqq \|Q(s, t)\| \leqq C(1+|s|)^{mp} \exp t\Lambda(s).$$

On classe les systèmes (1) suivant la croissance de $\Lambda(s)$: système parabolique si $\Lambda(\operatorname{Re} s) \leqq -C|\operatorname{Re} s|^h + C_1$ pour C, h, C_1 positifs convenables; hyperbolique si l'ordre de croissance exponentielle de Q est $\leqq 1$ et si $\Lambda(\operatorname{Re} s)$ est borné; correct au sens de Petrowsky si $\Lambda(\operatorname{Re} s)$ est borné. L'A. annonce les propriétés qu'il en déduit pour $G(x, t)$. Il en résulte l'existence et l'unicité du problème de Cauchy dans des espaces fonctionnels „composables en x" avec $G(x, t)$ [cf. Schwartz; Gel'fand-Šilov, loc. cit.].

J. L. Lions (Nancy).

Cabannes, Henri. Sur l'intégration d'une équation de Monge-Ampère. C. R. Acad. Sci. Paris **241** (1955), 1257–1259.

Verfasser behandelt und verallgemeinert die bereits von M. H. Martin [Canad. J. Math. **5** (1953), 37–39; MR **14**, 815] untersuchte Monge-Ampèresche Differentialgleichung

$$E \equiv \frac{\partial^2 z}{\partial \psi^2}\frac{\partial^2 z}{\partial p^2} - \Big(\frac{\partial^2 z}{\partial \psi\, \partial p}\Big)^2 + \delta(\psi)p^{2m-2} = 0.$$

Durch Transformation auf die von Stanjukovich zuerst verwendeten Variablen [Dokl. Akad. Nauk SSSR (N.S.) **96** (1954), 441–444; MR **16**, 301]

$$\psi_1 = \frac{1}{\psi},\ p_1 = \frac{p}{\psi},\ z_1 = \frac{z}{\psi},$$

ergibt sich aus $E=0$ die Differentialgleichung

$$E_1 \equiv \frac{\partial^2 z_1}{\partial \psi_1^2}\frac{\partial^2 z_1}{\partial p_1^2} - \Big(\frac{\partial^2 z_1}{\partial \psi_1 \partial p_1}\Big)^2 + \delta_1(\psi_1)p_1^{2m-2} = 0,$$

$$\delta_1 = \delta(\psi)\psi^{2m+2}.$$

Verfasser integriert die neue Gleichung für den Fall $\delta(\psi)=A\psi^{-2-2m}$ (A=Konstante$\neq 0$). Das Ergebnis kann benutzt werden, das Abklingen einer Stoßwelle in einer vollkommen kompressiblen Strömung zu berechnen.

M. Pinl (Köln).

Pompeiu, D. Un exemple de prolongement d'une solution à dérivées partielles. Com. Acad. R. P. Romîne **1** (1951), 391–393. (Romanian. Russian and French summaries)

If a solution of the equation

$$(*) \qquad (x-y)f_{xy} + f_x - f_y = 0$$

is given in $R-T$, where T is a triangle cut off of one corner of a rectangle R, the author extends this solution into the triangle T. He uses the fact that a solution of (*) also satisfies the equation $f_{zzyy}=0$ which he had considered in a previous paper.

D. L. Bernstein.

See also: Albrecht und Hochmuth, p. 467; Janet, p. 480; Pagni, p. 488; Pogorzelski, p. 497; Volkov, p. 539; Bergman, p. 549.

Difference Equations, Special Functional Equations

Zăgănescu, M. Sur une équation aux différences finies. Gaz. Mat. Fiz. Ser. A. **7** (1955), 315–321. (Romanian. Russian and French summaries)

The author considers the Möbius transformation

$$z' = [\alpha(\sigma)z + \beta(\sigma)]/[\bar{\beta}(\sigma)z + \bar{\alpha}(\sigma)].$$

When σ is a continuous real variable, the infinitesimal transformation of the group leads to a differential equation of Riccati's form. When σ is discontinuous, $\sigma=nh$, h a constant, n an integer, the Riccati equation has a finite-difference analogue which is here proved to be reducible to a linear difference equation of the second order. This last is a known result [see, e.g., Milne-Thomson, The calculus of finite differences, Macmillan, London, 1951, 11.8; MR **13**, 245]. An application is made to the discontinuous group $(z'+1)/(z'-1)=kh^n(z+1)/(z-1)$.

L. M. Milne-Thomson (Greenwich).

Davies, H. Poisson's partial difference equation. Quart. J. Math. Oxford Ser. (2) **6** (1955), 232–240.

For the problem of current flow in uniform media there are alternative idealizations. With the usual continuous model, one gets Poisson's equation, $\Delta^2 V = -4\pi\rho$. An alternative visualization as an infinite uniform regular grid of conducting wires leads to Poisson's partial difference equation

$$(1) \qquad 4V_{l,m} - V_{l-1,m} - V_{l+1,m} - V_{l,m-1} - V_{l,m+1} = RI\delta_{l0}\delta_{m0}$$

where δ_{rs} is the Kronecker delta. Equation (1) may be studied by direct analogy with continuous potential theory. Writing (in the case of two-dimensions)

$$\delta_{l0}\delta_{m0} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(lx+imy)}dxdy,$$

one seeks a solution of (1) in the form

$$V_{l,m} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y)e^{i(lx+imy)}dxdy$$

which leads directly to

$$V_{l,m} = -\frac{RI}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} \frac{1-\cos 2lx \cos 2my}{\sin^2 x + \sin^2 y}\,dxdy,$$

when one subtracts a suitable constant potential so as to obtain convergence at the origin. The author extends this idea in several directions, including computation of the total resistance between grid points, and application to higher dimensions.

The grid may be a direct realization of a physical grid, or an alternative to a continuous model. In the latter case, further questions arise concerning relations between alternative models. (A similar study of discrete harmonic solutions of Laplace's partial difference equation has been carried out by R.J. Duffin[Duke Eath. J. **20** (1953), 233–251; MR **16**, 1119].)

R. B. Davis (Durham, N.H.).

Križanovs'kiĭ, O. M. On quadratic criteria of the quality of transfer processes of regulation described by linear difference equations with constant coefficients. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 196–202. (Ukrainian. Russian summary)

Assume $\{u_k\}$ is a solution of the linear difference equation

$$u_{m+n}+a_1u_{m+n-1}+\cdots+a_nu_m=0 \quad (m=0,1,2,\cdots)$$

with the given initial values $u_0, u_1, \cdots, u_{n-1}$. Then the sums $I_n=\sum_{k=0}^{\infty} u_ku_{k+m}$ $(m=0,\cdots,n)$ satisfy a system of linear equations which is readily established. These systems are written out explicitly for the special cases $n=2, 3, \cdots, 6$. *M. Golomb* (Lafayette, Ind.).

★**Myschkis, A. D. Lineare Differentialgleichungen mit nacheilendem Argument.** Deutscher Verlag der Wissenschaften, Berlin, 1955. ix+181 pp. DM 21.30.

Translation of Myškis's Lineĭnye differential'nye uravneniya s zapazdyvayuščim argumentom [Gostehizdat, Moscow, 1951; MR **14**, 52].

Hahn, Wolfgang. Über analytische Lösungen linearer Differential-Differenzengleichungen. Math. Z. **63** (1955), 313–319.

The author investigates series representations of analytic solutions of the equation

$$(1) \qquad \sum_{i=0}^{p} a_iY^{(i)}(t)+\sum_{i=0}^{q} b_iY^{(i)}(t-1)=0,$$

$p>q$, where a_i, b_i are (complex) constants. Such series are built up from particular solutions of the form e^{r_it}, where $\{r_i\}$ are the zeros of the characteristic equation

$$(a) \qquad A(r)=\sum_{i=0}^{q} a_ir^i+e^{-r}\sum_{i=0}^{q} b_ir^i=0.$$

In particular, series of the form (b) $f(z)=\sum_{n=1}^{\infty} A_ne^{s_nz}$ are considered where $\{s_n\}$ are those (simple) zeros of $A(r)$ with a positive imaginary part. (Similar results hold for negative imaginary parts.)

Such series converge in a half-plane bounded by a horizontal convergence line: Im $z=a$. The results include the determination of a; the determination of a ray of convergence and a ray of divergence on the convergence line; and theorems of Abelian and Tauberian type with respect to a point on the convergence line.

I. M. Sheffer (University Park, Pa.).

Davidenko, D. F. On application of the method of variation of parameters to the theory of nonlinear functional equations. Ukrain. Mat. Ž. **7** (1955), 18–28. (Russian)

Let G be a bounded closed region in n-dimensional space, E the space of functions $\varphi(x)$ defined for $x \in G$ with norm $|\varphi|=\sup_{x\in G}|\varphi(x)|$ and $K(x,\varphi)$ a nonlinear functional on E having a differential $\delta K(x,\varphi)=\int_G Q(x,x',\varphi)\,\delta\varphi(x')dx'$, where $Q(x,x',\varphi)$ is a functional of $\varphi \in E$ depending on $x, x' \in G$. In order to solve the equation (1) $\varphi(x)+\lambda K(x,\varphi)=0$, where λ is a real parameter, the author considers the solution as a function of λ, $\varphi=\varphi(x,\lambda)$, and uses the differentiated equation

$$\frac{\partial}{\partial\lambda}\varphi(x,\lambda)+\lambda\int_G Q(x,x',\varphi)\,\frac{\partial}{\partial\lambda}\varphi(x',\lambda)d\lambda=-K(x,\varphi)$$

which is linear in $\partial\varphi/\partial\lambda$. Its solution, given by the Fredholm resolvent, has the form $\varphi(x,\lambda)=\int_0^{\lambda} I(x,\lambda,\varphi(x,\lambda))d\lambda$. This functional equation, and with it (1), are shown to have a unique solution $\varphi(x,\lambda)$ satisfying $|\varphi|\leq L$ for all λ, $|\lambda|\leq\lambda^*$, where λ^* is an explicitly given bound, provided the Fredholm determinant $D(\lambda,\varphi)$ of the kernel $Q(x,x',\varphi)$ satisfies $|D(\lambda,\varphi)|\geq a$ for $|\varphi|\leq L$, $x \in G$, $|\lambda|\leq d$, where L, a, d are positive constants. *M. Golomb* (Lafayette, Ind.).

Steinhaus, H. On a certain power series. Prace Mat. **1** (1955), 276–284. (Polish. Russian and English summaries)

The function $f(x)=\sum_{n=0}^{\infty}(-1)^nx^{2^n}$ is the only analytic solution of the functional equation $f(x)+f(x^2)=x$ $(0\leq x<1)$. There are infinitely many other solutions, even continuous in $0<x<1$, but none of them continuous at $x=0$. The author does not know a solution continuous at the point 1 and states some other problems concerning f and related functions. *K. Zeller* (Tübingen).

See also: Krishna Iyer, p. 499; Rådström, p. 509.

Integral Equations, Equations in Infinitely Many Variables

★**Krein, M. G. On a new method of solving linear integral equations of the first and second kinds.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 6 pp. $3.00.

Translated from Doklady Akad. Nauk SSSR (N.S.) **100** (1955), 413–416; MR **17**, 375.

Pogorzelski, W. Sur le système d'équations intégrales à une infinité de fonctions inconnues. Ann. Polon. Math. **2** (1955), 106–117.

The system of equations is

$$\varphi_n(x)=\lambda\int_{\Omega} F_n(x,y,\varphi_1(y),\cdots,\varphi_n(y),\cdots)/|xy|^{\alpha_n}\,dy \qquad (n=1,2,\cdots),$$

where Ω is a closed bounded subset of Euclidean space of s dimensions, $|xy|$ is the distance in this space, $\alpha_n\leq\alpha<s$, $F_n(x,y,\ ;u_1,\cdots,u_n,\cdots)$ are continuous and uniformly bounded on S: x, y on Ω, $|u_i|\leq R$. Continuous solutions $\varphi_n(x)$ subject to the condition $|\varphi_n(x)|\leq R$ are sought. The existence of a solution for sufficiently small λ is reduced to the Schauder fixed-point theorem [Studia Math. **2** (1930), 171–180, p. 175] by introducing a metric on S:

$$(A,A')=|xx'|+|yy'|+\sum_n \varepsilon_n|u_n-u_n'|$$

with $\varepsilon_n>0$, $\sum_n\varepsilon_n<\infty$, and a norm on the linear space of sequences $U=\{\varphi_n(x)\}$ of continuous uniformly bounded functions, viz. $\|U\|=\sum_n\varepsilon_n\sup_x|\varphi_n(x)|$. The integrals on the right of the equations then define a continuous transformation on $|\varphi_n(x)|\leq R$ to a compact subset of the same set. The result is applied to the solution of a system of partial differential equations $\Delta u_n=\lambda F_n(A,u_1,\cdots,u_n,\cdots)$ subject to the boundary conditions $u(A_n)\to 0$ as A_n approaches the boundary of the three-dimensional domain Ω. If the F_n satisfy a Hölder conditions:

$$|F_n(A,u_1,\cdots,u_m,\cdots)-F_n(A',u_1',\cdots,u'_m,\cdots)|< k_n|AA'|^{\mu}+\sum_m \varepsilon_{nm}|u_m-u_m'|^{\mu}$$

with $0<\mu<1$, then via Green's functions the solution of these equations are equivalent to a system of integral equations. A system of Volterra integrals equations on the straight line is also considered. *T. H. Hildebrandt.*

Vasil'ev, V. V. Solution of Cauchy's problem for a class of linear integro-differential equations. Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 849–852. (Russian)

The integro-differential equation considered in this paper is the same as that studied by the author in a previous paper [Prikl. Mat. Meh. **15** (1951), 609–614; MR **13**, 354]. The Cauchy problem where the initial data $z(x_0), \cdots, z^{(n-1)}(x_0)$ are prescribed, leads to an algebraic system of linear equations for the constants c_k in $z=c_1z_1(x)+\cdots+c_nz_n(x)$ which is part of the general solution found in the earlier paper. *M. Golomb.*

Èbanoidze, T. A. On a class of nonlinear integral equations. Soobšč. Akad. Nauk Gruzin. SSR **15** (1954), 7–12. (Russian)

The author proves the existence of a unique solution $y \in Lq$ $(p>1)$ for the equation

$$y(x)=f(x, \lambda \int_G K(x, t, y(t))dt),$$

where G is a cube in n-dimensional space, for all sufficiently small real λ, provided $f(x, y)$, $K(x, t, y)$ satisfy, as functions of y, certain Lipschitz conditions which make the solution appear as the fixed point of a contracting transformation. *M. Golomb* (Lafayette, Ind.).

Povolockiĭ, A. I. Nonlocal existence theorems for solutions of systems of nonlinear integral equations. Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 901–904. (Russian)

If B is a completely continuous operator in a sphere T of a Hilbert space which includes its 0 point and if $(B\varphi, \varphi)<(\varphi, \varphi)$ for all φ on the surface of T then $B\varphi=\varphi$ has at least one solution in T. Using this fixed-point theorem of Krasnosel'skiĭ [same Dokl. (N.S.) **88** (1953), 949–952; MR **14**, 1095] the author establishes existence of a solution for the system of integral equations

$$\varphi_i(x)=\lambda \int_G K_i(x, y)f_i[y, \varphi_1(y), \cdots, \varphi_n(y)]dy \quad (i=1, \cdots, n)$$

under various conditions. A typical set of conditions is: The mapping

$$A: \{u_1(x), \cdots, u_n(x)\} \to \left\{\int_G K_1(x, y)u_1(y)dy, \cdots, \int_G K_n(x, y)u_n(y)dy\right\}$$

can be factored, $A=HH^*$, with H completely continuous;

$$f_k(x, z_1, \cdots, z_n)z_k \leqq a_kz_k^2+b_k(x)|z_k|^{2-\gamma_k}+c_k(x);$$

$$\lambda<\min(1/a_i\lambda_i), \quad i=1, \cdots, n,$$

where λ_i is the largest eigenvalue of $K_i(x, y)$. A few theorems concerning the factoring $A=HH^*$ and the mapping

$$\{u_1(x), \cdots, u_n(x)\} \to \{f_1(x, u_1(x), \cdots, u_n(x)), \cdots, f_n(x, u_1(x), \cdots, u_n(x))\}$$

are also stated. No proofs are given. *M. Golomb.*

★ **Akhiezer, N. I. On some coupled or dual integral equations.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 6 pp. $3.00. Translated from Doklady Akad. Nauk SSSR (N.S.) **98** (1954), 333–336; MR **16**, 487.

Koiter, W. T. Approximate solution of Wiener-Hopf type integral equations with applications. I. General theory. Nederl. Akad. Wetensch. Proc. Ser. B. **57** (1954), 558–564.

Koiter, W. T. Approximate solution of Wiener-Hopf type integral equations with applications. IIa. Some plate bending problems and an example from hydrodynamics. Nederl. Akad. Wetensch. Proc. Ser. B. **57** (1954), 565–574.

Koiter, W. T. Approximate solution of Wiener-Hopf type integral equations with applications. IIb. Some plate bending problems and an example from hydrodynamics. Nederl. Akad. Wetensch. Proc. Ser. B. **57** (1954), 575–579.

The author discusses a method whereby the Fourier transform of the kernel of an integral equation of the Wiener-Hopf type may be approximated in its strip of regularity. The reason for such a procedure is to simplify the factoring operation and hence this remark is to be kept in mind in the approximation work. Following a discussion of this operation, the author applies this to the solution of the following boundary-value problems. Solve the partial differential equation

$$\Delta\Delta w=0$$

subject to the boundary conditions

$$(*) \quad w=\frac{\partial^2 w}{\partial y^2}=0, \quad -\infty<x<\infty, \quad y=1,$$

$$w=\frac{\partial^2 w}{\partial y^2}=0, \quad x<0, \quad y=0,$$

$$w=0, \quad \frac{\partial^2 w}{\partial y^2}=-m(x), \quad x>0, \quad y=0,$$

and $m(x)$ is to be determined from other auxiliary conditions.

The second problem replaces (*) by

$$w=\frac{\partial w}{\partial y}=0, \quad -\infty<x<\infty, \quad y=1.$$

Numerical results are indicated and it is shown that the approximation method leads to results which agree well with the exact solutions. *A. E. Heins.*

Koiter, W. T. Approximate solution of Wiener-Hopf type integral equations with applications. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 257–258.

Abstract of the papers reviewed above.

Urbanik, K. On a certain infinite system of equations. Prace Mat. **1** (1955), 253–255. (Polish. Russian and English summaries)

For every pair of sequences of positive integers $\{m_k\}$ and $\{n_l\}$ such that $m_k \neq n_l$ $(k, l=1, 2, \cdots)$ and for every A such that $|A|>1$ the system of equations

$$\sum_{k=1}^{\infty} x_k/(n_l^2-m_k^2)=Ax_l \quad (l=1, 2, \cdots)$$

has, among the bounded sequences, only the zero solution. The proof depends on the fact, that the transformation defined by $y_l=A^{-1}\sum_k x_k/(n_l^2-m_k^2)$ $(l=1, 2, \cdots)$ is a contraction in the space m of the bounded sequences (with the usual norm). *K. Zeller* (Tübingen).

Hartman, Philip, and Wintner, Aurel. The spectra of Toeplitz's matrices. Amer. J. Math. 76 (1954), 867–882.

Ist $f(\varphi) \in L^1(-\pi, \pi)$, $f(\varphi) \sim \sum_{n=-\infty}^{+\infty} f_n e^{in\varphi}$, so heißt $\mathfrak{T}=(f_{n-m})$ $(n, m=0, 1, 2, \cdots)$ die zu $f(\varphi)$ gehörende Toeplitzsche Matrix ($\mathfrak{T}$-Matrix). Ist $f(\varphi) \in L^2$, $x^+(\varphi) \in L^2$, $x^+(\varphi) \sim \sum_{n=0}^{\infty} x_n e^{in\varphi}$ (d.h. $x^+(\varphi)$ innere Funktion), $x^+=(x_0, x_1, x_2, \cdots)$, so ist $\mathfrak{T}x^+=y^+$ äquivalent mit $f(\varphi)x^+(\varphi)=y(\varphi) \in L^1$, $y(\varphi) \sim \sum_{n=-\infty}^{+\infty} y_n e^{in\varphi}$, $y^+=(y_0, y_1, y_2, \cdots)$. Ist $f(\varphi)$ reell, so $\mathfrak{T}$ hermitesch; in diesem Falle ist $\mathfrak{T}$ genau dann beschränkt, wenn $f(\varphi)$ (wesentlich) beschränkt ist. I. Beschränkte $\mathfrak{T}$-Matrizen: $f(\varphi)$ sei reell, meßbar und (wesentlich) beschränkt; m, M seien (wesentliche) untere und obere Grenze von $f(\varphi)$. Dann ist das Spektrum von $\mathfrak{T}$ das abgeschlossene Intervall $[m, M]$; ist $f(\varphi)$ nicht konstant $(m<M)$, so ist das Punktspektrum von $\mathfrak{T}$ leer. II. Riemanns Problem: Sei $f(\varphi)$ reell, meßbar und (wetentlich) beschränkt, dann sind notwendig für die Existenz einer Lösung $x_0^+(\varphi) \in L^2$, $x_0^+(\varphi) \sim \sum_{n=0}^{\infty} x_n e^{in\varphi}$ von

$$(*)\quad f(\varphi)x_0^+(\varphi)=1+y_0^-(\varphi)\ [y_0^-(\varphi) \in L^2, y_0^-(\varphi) \sim \sum_{n=1}^{\infty} y_{-n} e^{in\varphi}]$$

die beiden Bedingungen (a) $f(\varphi)$ hat konstantes Vorzeichen, (b) $1/f(\varphi) \in L^1$. Sei $f(\varphi) \in L^1$. Dann sind (a), (b) für die Lösbarkeit von (*) durch eine innere Funktion $x^+(\varphi) \in L^2$ hinreichend [es ist dann $y^-(\varphi) \in L^2$]. III. Gegenbeispiel: Sei $f(\varphi) \in L^2$ und reell. Dann ist es möglich, daß $\lambda=0$ zum Punktspektrum von $\mathfrak{T}$ gehört, und ferner, daß (*) eine Lösung $x^+(\varphi) \in L^2$ besitzt, ohne daß (a) gilt. IV. Unbeschränkte $\mathfrak{T}$-Matrizen: Sei $f(q) \in L^2$ und reell. 1. Die für beschränktes f gemachten drei Aussagen (I. und die erste Aussage von II.) gelten, falls $\mathfrak{T}$ selbstadjungiert ist. 2. $\mathfrak{T}$ ist selbstadjungiert, falls $f(\varphi)$ halbbeschränkt ist. 3 (auch gültig, falls $\mathfrak{T}$ nicht selbstadjungiert ist). Seien $m(\geqq -\infty)$ und $M(\leqq +\infty)$ (wesentliche) untere und obere Schranke von $f(\varphi)$, $m<\mu<M$, J die Einheitsmatrix, so besitzt $\mathfrak{T}-\mu J$ keine beschränkte Inverse. Anhang: Die $\mathfrak{T}$-Matrix einer beliebigen (auch nicht reellen) Funktion $f(\varphi) \in L^2$ ist genau dann beschränkt, wenn $f(\varphi)$ (wesentlich) beschränkt ist. *F. W. Schäfke* (Zbl 56, 113).

See also: Albertoni, p. 474; Križanovs'kiĭ, p. 488; Davidenko, p. 497; Warschawski, p. 540; Krook, p. 543.

Calculus of Variations

Faulkner, Frank. A degenerate problem of Bolza. Proc. Amer. Math. Soc. 6 (1955), 847–854.

The problem is to minimize a given function $f(x_2, y_2, z_2)$ in the class of piecewise-smooth curves in space joining the points (x_1, y_1, z_1), (x_2, y_2, z_2) and satisfying a differential equation of the form $dz=K(x, y, z)dx$ as well as the inequality $dx/ds \geqq 0$ (s denotes arc length). The endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2) are variable on manifolds M_1, M_2 respectively. It seems to the reviewer that the author's necessary conditions are valid only under some additional hypothesis not stated which would insure that $f(x_2, y_2, z_2)$ is minimum when z_2 is minimum.

W. H. Fleming (Lafayette, Ind.).

★ **Vazsonyi, Andrew. Optimizing a function of additively separated variables subject to a simple restriction.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 453–469. National Bureau of Standards, Washington, D.C., 1955.

Consider the non-negative variables $x_1, x_2, \cdots, x_N$, and the functions $f_1(x_1), f_2(x_2), \cdots, f_N(x_N)$. The problem is to maximize $\sum_{i=1}^{N} f_i(x_i)$ subject to $\sum_1^N x_i \leqq A$. In order to solve this problem, consider the problem of maximizing $\sum_{i=1}^{n} f_i(x_i)$ $(n=1, 2, \cdots, N)$ subject to $\sum_{i=1}^{n} x_i \leqq a$. Let this maximum be denoted by $F_n(a)$. Then $F_1(a)=\max_{0\leq x\leq a} f_1(x)$ and $F_n(a)$ can be computed from the algorithm $F_n(a)=\max_{0\leq x\leq a}[F_{n-1}(a-x)+F_1(x)]$.

When the $f_i(x)$ functions are convex polygons, the problem can be formulated as a linear-programming problem. However, the algorithm given here gives a very simple direct solution for this case. *T. L. Saaty.*

See also: Miranda, p. 493; Skornyakov, p. 522; Fel'dbaum, p. 563.

Theory of Probability

Takács, Lajos. On a general theorem of probability theory. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 467–476. (Hungarian)

The author proves in several ways the well-known formula for the probability $P_{[k]}$ of the realization of exactly k among the n events $A_1, A_2, \cdots, A_n$. His proofs are based on the fact that the quantity

$$S_k=\sum P(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$$

can be interpreted as the binomial moment $E\left\{\binom{Y}{k}\right\}$ of the random variable Y which represents the number of realized events. He states that this approach was already pointed out by Ch. Jordan [Bull. Soc. Math. France 67 (1939), 223–242; MR 1, 340]. The particular case of equivalent events as well as various applications are also discussed. *E. Lukacs* (Washington, D.C.).

Blum, J. R. A note on estimating distribution functions. Proc. Amer. Math. Soc. 6 (1955), 953–957.

Let $f_n(y)$ be a sequence of continuous functions, and $F(y)$ a distribution function on $[-A, A]$. The author considers the problem of estimating $F(y)$ from the sequence $a_n=\int_{-A}^{A} f_n(y)dF(y)$. He shows that a necessary and sufficient condition for $F(y)$ to be uniquely determined by $\{a_n\}$ is that $f_n(y)$ be fundamental in the Banach space of continuous functions on $[-A, A]$ and that if this condition is satisfied a sequence of step functions $F_n(y)$ can be constructed such that $\lim_{n\to\infty} F_n(y)=F(y)$ at every point of continuity of $F(y)$. *J. Blackman.*

Krishna Iyer, P. V. The use of difference equation in solving distribution problems. Bull. Inst. Internat. Statist. 23 (1951), part II, 97–104.

A number of distributions can be obtained when points of s characters are distributed at random on a lattice of points consisting of m rows and n columns. Some of these distributions are quite complicated and some authors have used the method of difference equations to obtain their probability generating functions (p.g.f.). The difference equations satisfied by the p.g.f.'s enable one to obtain the actual generating functions by either soiving the equations or by using them in succession after evaluating the p.g.f.'s for some small values of m and n. In this paper, the author has obtained the difference equations satisfied by the p.g.f.'s of the distributions for the number of (i) runs of length l, (ii) triplets, (iii) junctions between

ascending, descending, and stationary runs, (iv) total number of positive or negative changes of signs in blocks of three observations and (v) joint distribution for runs of length k and l for a set of observations in a sequence. He has also obtained the difference equation for the distribution of joins between points of two characters for a lattice along the two perpendicular axes.

O. P. Aggarwal (Lafayette, Ind.).

Méric, Jean. Sur une expression de la fonction $K(x, y)$ de Pólya, liée au test de Wald. C. R. Acad. Sci. Paris **241** (1955), 1255–1257.

Let T denote the set of lattice points (x, y) in the first quadrant satisfying $-k_1<ay-bx<k_2$ where k_1, k_2, a, b are positive integers with $a\leq b$. $K(x, y)$ denotes the number of ways of reaching the point (x, y) from the origin by successive unit motions in the positive x or y direction without leaving T. The author writes an expression for the value of $K(x, y)$ in terms of the values of K at certain other points. *J. Kiefer* (Ithaca, N.Y.).

Méric, Jean. Sur la relation de récurrence de Pólya liée au test de Wald. C. R. Acad. Sci. Paris **241** (1955), 1377–1380.

Equating the function K of the preceding review to zero at a certain set of lattice-boundary points of T implies the vanishing of a determinant, which fact the author will use in a later paper to help calculate K.

J. Kiefer (Ithaca, N.Y.).

Bobrov, A. A. Remark on distribution functions of regular growth. Vestnik Leningrad. Univ. **10** (1955), no. 11, 65–67. (Russian)

A distribution function $F(x)$ is said to be of regular growth as $x\to\infty$ if

$$\lim_{x\to\infty}\frac{1-F(cx)}{1-F(x)}=f(c)$$

exists for all $c>0$; here $f(c)$ may be $+\infty$. The author proves by elementary methods that always $f(c)=c^{-\alpha}$, where $0\leq\alpha\leq+\infty$ and α depends on F. Similar results hold when $x\to-\infty$. *J. Wolfowitz* (Ithaca, N.Y.).

Simaika, Jacques. Sur une valeur typique d'une distribution de directions. C. R. Acad. Sci. Paris **241** (1955), 1375–1377.

Let θ be a random angle and write

$$\mathfrak{M}\varphi(\theta)=\int\varphi(\beta)\Pr(\theta\in d\beta).$$

Given $\alpha\in[0, 2\pi)$ and setting $\theta_\alpha=(\theta-\alpha) \bmod (2\pi)$, we have $\mathfrak{M}\theta_\alpha=\mathfrak{M}\theta-\alpha+2\pi\Pr(\theta\leq\alpha)$. The author proposes to avoid this inelegant formula by using the typical direction

$$\tau\theta=\text{argument }(\mathfrak{M}e^{i\theta}),\ |\mathfrak{M}e^{i\theta}|>0,$$

in place of the expectation $\mathfrak{M}\theta$. One has $\tau(\theta_1+\theta_2)=(\tau\theta_1+\tau\theta_2) \bmod (2\pi)$ when θ_1 and θ_2 are independent, so that $\tau\theta_\alpha=(\tau\theta-\alpha) \bmod (2\pi)$, and, in addition, the typical value minimizes the expected distance

$$\Delta(\beta)=\mathfrak{M}|e^{i\theta}-e^{i\beta}|^2.$$

H. P. McKean, Jr. (Princeton, N.J.).

Špaček, Antonín. Regularity properties of random transforms. Czechoslovak Math. J. **5(80)** (1955), 143–151. (Russian summary)

Given a probability measure space $(F, \mathfrak{F}, \mu)$ and a set E in F, J. L. Doob [Bull. Amer. Math. Soc. **53** (1947), 15–30; MR **8**, 472] has remarked that there exists a probability measure ν on the Borel algebra $\mathfrak{E}=(B:B=A\cap E\ A\in\mathfrak{F})$ such that $\nu(A\cap E)=\mu(A)\Leftarrow\Rightarrow E$ has outer (μ) measure one. The new probability measure space $(E, \mathfrak{E}, \nu)$ is called a standard extension.

In the present paper, F is the space of maps from one abstract space to another, $\mathfrak{F}$ is the Borel extension of certain finite-dimensional cylinder sets, and standard extensions are used to show that, in a wide variety of circumstances, one can concentrate on the smooth maps.

H. P. McKean, Jr. (Princeton, N.J.).

Erdös, Paul, and Offord, A. C. On the number of real roots of a random algebraic equation. Proc. London Math. Soc. (3) **6** (1956), 139–160.

The authors refine the results of Littlewood and Offord [Proc. Cambridge Philos. Soc. **35** (1939), 133–148], who gave estimates of the expected number of real roots for an equation of degree n when the coefficients are identically distributed random variables. In particular, they studied the family of equations

$$f_n(x)=1+\varepsilon_1x+\varepsilon_2x^2+\cdots+\varepsilon_nx^n=0 \tag{1}$$

where $P(\varepsilon_\nu=1)=\frac{1}{2}$ and $P(\varepsilon_\nu=-1)=\frac{1}{2}$ for $\nu=1, 2, \cdots, n$. Erdös and Offord prove that the number of real roots of most of the equations of family (1) is

$$\frac{2}{\pi}\log n+o\{(\log n)^{\frac{1}{2}}\log(\log n)\}, \tag{2}$$

and that the exceptional set does not exceed a proportion

$$o\{(\log\log n)^{-\frac{1}{2}}\}$$

of the total number of equations. Even for small n their asymptotic results are reasonably good. For example, $2\pi^{-1}\log n=1.324$ when $n=8$. K. H. V. Booth [Math. Tables Aids Comput. **8** (1954), 184–186] has shown that of the 256 equations

$$1\pm x\pm x^2\pm\cdots\pm x^8=0,$$

58 have no real roots, 190 have 2 real roots, 8 have 4 real roots, and none have more than 4, giving an average of 1.609. *H. P. Edmundson* (Santa Monica, Calif.).

Mohan, C. The gambler's ruin problem with correlation. Biometrika **42** (1955), 486–493.

Peter and Paul toss coins, with Peter winning one unit with heads and losing one unit with tails, as follows. After any toss in which Peter wins (loses) the coin used for the next toss has probability p_1 (q_2) of heads. The probability of Peter's ultimate ruin when his initial capital is r and Paul's is $\alpha-r$ is found to be

$$[c_1q_1(\lambda^{\alpha-1}-\lambda^r)+c_2(q_1\lambda^{\alpha-1}-q_2\lambda^{r-1})]/(q_1\lambda^{\alpha-1}-q_2)$$

for $\lambda=p_2/p_1\neq1$, where $q_1=1-p_1$, $p_2=1-q_2$ and $c_1(c_2=1-c_1)$ is the probability that Peter wins the initial game, and $[c_2+(\alpha-1-r)q]/[1+q(\alpha-2)]$ for $q_2=q_1=q$. In the latter case, Peter's chance as a function of the correlation $R=p-q$ between the outcomes of successive games ($c_1=c_2=\frac{1}{2}$) is investigated. The chances of the player with smaller initial capital increase with the correlation. Explicit formulas for the generating function of the duration of play and for the expected duration are obtained. *D. Blackwell* (Berkeley, Calif.).

Has'minskiĭ, R. Z. Probability distribution of functionals of trajectories of a random process of diffusion type. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 22–25. (Russian)

Let $\{X(t), t\geqq 0\}$ be a one-dimensional Markov process, with initial point $x \in (r_1, r_2)$. It is supposed that the transition probability densities are fundamental solutions of the backward diffusion equation $u_t = bu_{xx} + au_x$, where a, b do not depend on t and a, b' are continuous on (r_1, r_2), with $b>0$. Let $\tau(x)$ be the time a trajectory first meets r_1 or r_2 (taken as $+\infty$ if there is no such meeting). The author investigates the distribution of $\zeta(x) = \int_0^{\tau(x)} V[X(t)]dt$, where V is continuous on (r_1, r_2) or has at worst isolated discontinuities of the first kind there. Let ζ_0 be ζ with V replaced by $|V|$. Let $W = \exp(-\int_{a_0}^{a}(a/b)dy)$, and let $R = W(x)\int_{a_0}^{a}|V|\, b^{-1}W^{-1}dy$. The point r_j is called attracting if, with positive probability, (*) $X(t)\to r_j$ when $t\uparrow\tau(x)$. The following results are stated without proof. (1) The point r_j is attracting if and only if W is integrable in a neighborhood of r_j. This result has also been obtained by the reviewer [Trans. Amer. Math. Soc. **78** (1955), 168–208; MR **17**, 50]. (2) For almost all trajectories for which (*) is true, $\zeta_0(x)<\infty$ if R is integrable in a neighborhood of r_j. If R is not so integrable, then $\zeta_0(x)=\infty$ for almost all trajectories for which (*) is true. (3) If at least one of the points r_j is attracting, and of R is integrable in a neighborhood of each attracting point, then $\zeta_0(x)<\infty$ with probability 1, and $E\{\exp(is\zeta(x))\}$ (s real) is the unique bounded solution of (**) $bz''+az'+isVz=0$ with boundary value 1 at each attracting point. (The equation is to hold at continuity points of V and z, z' are to be continuous even at the discontinuities of V.) If $Q_s^{(r_j)}(x)$ is the above expected value except that the exponential is defined as 0 unless (*) is true, then, if r_1 is attracting, and if R is integrable in a neighborhood of r_1, $Q_s^{(r_1)}$ is the unique bounded solution of (**) with boundary value 1 at r_1 and, if r_2 is also attracting, the boundary value 0 at r_2. (4) Differential equations satisfied by $E\{(\int_0^{\tau(x)} V[X(t)]dt)^n\}$ and the modified expectation with the integral replaced by 0 when (*) is not true are given. (5) Let r_1 be attracting, and let $\tilde{Q}^{(r_1)}(x)$ be defined like $Q_s^{(r_1)}(x)$, but with $s=-i$. Then $\tilde{Q}^{(r_1)}(x)<\infty$ if and only if (**), with $s=-i$, has a solution not vanishing in $[r_1, r_2)$, and if so $\tilde{Q}^{(r_1)}(x)$ is (A) the unique solution of (**) with $z(r_1)=1-z(r_2)=1$ if both endpoints are attracting, or (B) the least positive solution with $z(r_1)=1$ if r_2 is not attracting.

Various applications are made. For example, if $V>0$, the differential equation found for the Laplace transform of the distribution of ζ can be used to evaluate this distribution. The distribution for a Brownian motion process, of the time a trajectory from a point spends to the right of the point before reaching a specified point to its left is evaluated.

Similar but more special results have been obtained for constant V by Darling and Siegert [Ann. Math. Statist. **24** (1953), 624–639; MR **15**, 449], Feller [Trans. Amer. Math. Soc. **77** (1954), 1–31; MR **16**, 150], and Doob [loc. cit.]. In the special case of Brownian-motion processes, see also Kac, ibid. **65** (1949), 1–13 [MR **10**, 383], Rosenblatt, ibid. **71** (1951), 120–135 [MR **13**, 258], and Ray, ibid. **77** (1954), 299–321 [MR **16**, 593]. Dynkin [see the paper reviewed below] studies this general problem by semigroup methods in a more general setting in which the state space of the process may be any metric space.

J. L. Doob (Urbana, Ill.).

Dynkin, E. B. Functionals of trajectories of Markov random processes. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 691–694. (Russian)

Let an $x(t)$ process be a separable measurable Markov process, with stationary transition probabilities and state space a metric space $\mathcal{E}$. In the following C is the Banach space of bounded continuous functions on $\mathcal{E}$, and $f_n \Rightarrow f$ means f_n, $f \in C$ and pointwise and bounded convergence of the function sequence. It is supposed that

(A) $$P\{x(t) \in U \mid x(0)=x\}\to 1 \text{ when } t\to 0,$$

if U is any neighborhood of x, and (B) $E\{f[x(t)] \mid x(0)=x\}$ defines a continuous function $T_t f$ of x whenever $f \in C$. The T_t family of transformations is then a semigroup, and, if A is defined by the relation $(T_t f - f)/t \Rightarrow Af$ ($t\to 0$) whenever this limit exists, A is called the infinitesimal generator of the semigroup. Under the stated hypotheses, (a) $T_t f \Rightarrow f$ when $t\to 0$, (b) $f_n \Rightarrow f$ implies that $T_t f_n \Rightarrow f$, (c) $\|T_t\|\leq 1$. If two Markov processes have the same infinitesimal generators, they have the same transition probabilities. If a semigroup operating on C satisfies (a)–(c), and has infinitesimal generator A, then, if Af is defined, $T_t f = u_t$ satisfies the equation $du_t/dt = Au_t$, and is the only solution of this equation with $u_t \Rightarrow f$ for $t\to 0$. If $V \in C$ and if $\zeta(0, t) = \int_0^t V[x(\tau)]d\tau$, $(S_t f)(x) = E\{f[x(t)] \exp \zeta(0, t) \mid x(0)=x\}$, then S_t defines a semigroup on C satisfying (a), (b) and $\|S_t\|\leq \exp(\|V\|t)$. The infinitesimal operator has the same domain as that of the T_t semigroup and is $A+V$. The characteristic function $\phi(\lambda, t, x) = E\{\exp i\lambda\zeta(0, t) \mid x(0)=x\}$ is the unique solution of the equation $d\phi/dt = A\phi + i\lambda V\phi$ for which $\phi \Rightarrow 1$ when $t\to 0$, and for which $\phi \exp(-\|V\|t)$ is bounded when $t\to\infty$.

See the preceding review for related work and references. The results are based to some extent on calculations involving a more general situation in which V may depend also on t. *J. L. Doob* (Urbana, Ill.).

Hajnal, J. The ergodic properties of non-homogeneous finite Markov chains. Proc. Cambridge Philos. Soc. **52** (1956), 67–77.

Let $P_k = \{{}_k p_{ij}\}$, $k=1, 2, \cdots; i=1, \cdots, N; j=1, \cdots, N$, be stochastic matrices, i.e., square matrices of non-negative elements such that $\sum_{j=1}^{N} {}_k p_{ij}=1$ identically in k and i. The author studies the asymptotic (with n) behavior of $\prod_{k=1}^{n} P_k$. Let $\pi_k = \min_{i,j}({}_k p_{ij}, i, j=1, \cdots, N)$. Typical results are: Theorem 2: If $\prod_{k=1}^{\infty}(1-\pi_k)=0$, then $\lim_{n\to\infty}\max_{i_1,i_2,j}|{}_n l_{i_1 j} - {}_n l_{i_2 j}|$, where ${}_n l_{ij}$ is the element in the ith row and jth column of $\prod_{k=1}^{n} P_k$. Theorem 1: If $\prod_{k=1}^{\infty}(1-\pi_k)=0$ and there exists a sequence of stochastic matrices S_k, $k=1, 2, \cdots$ with identical rows such that a) $\sum (S_k P_k - S_{k+1})$ converges absolutely, b) $S_k \to S$, then $\prod_{k=1}^{\infty} P_k = S$. *J. Wolfowitz* (Ithaca, N.Y.).

Eberl, W. Die Summenverteilung verketteter Alternativen. Österreich. Ing.-Arch. **9** (1955), 280–288.

Let $(x_n : n>0)$ be a Markov chain with two possible states, 0 and 1, and stationary transition probabilities. Then, setting $S_n^* = n^{-\frac{1}{2}}\sum_{0<i\leq n}(x_i - E(x_i))$ and making $n\uparrow+\infty$, $E(S_n^*)^2 \to$ some number σ^2, and

$$\Pr(S_n^* < s) \to (2\pi\sigma^2)^{-\frac{1}{2}}\int_{x<s} \exp(-x^2/2\sigma^2)dx$$

[A. Markov, Acta Math. **33** (1909), 87–104]. The author presents a simple combinatorial proof and shows, by a numerical example, that the Gaussian approximation is very good, even for small n. *H. P. McKean, Jr.*

Gihman, I. I. On the theory of differential equations of stochastic processes. I, II. Amer. Math. Soc. Transl. (2) **1** (1955), 111–137, 139–161.

Translated from Ukrain. Mat. Ž. **2** (1950), no. 4, 37–63; **3** (1951), 317–339; MR **14**, 61, 1101.

Smith, W. L. Regenerative stochastic processes. Proc. Roy. Soc. London. Ser. A. **232** (1955), 6–31.

Let $\{y_n\}$ $(n=0, 1, 2, \cdots)$ be nonnegative independent chance variables such that the y_n for $n>0$ have a common aperiodic distribution, and for any $t>0$ let T_t be the largest of the sums $y_0+\cdots+y_n$ which does not exceed t. A process $\{X_t\}$ is called an equilibrium process with respect to a family $\mathfrak{A}$ of sets in the range of X_t if there is a function $\phi_A(t)$ such that $\Pr\{X_t \in A | T_t, T_t>0\}=\phi_A(t-T_t)$. Under alternative regularity conditions, e.g., the distribution of $y_1+\cdots+y_n$ has an absolutely continuous component for some n, $\phi_A(t)$ is measurable, and $\mu_1=Ey_1<\infty$, it is shown that

$$\Pr\{X_t \in A | X_0\} \to \int_0^\infty \phi_A(v)[1-F(v)]dv/\mu,$$

as $t\to\infty$ for all $A \in \mathfrak{A}$. This and related results enable the author to extend known results on the approach to a stationary distribution of Markov processes with a countable number of states to processes which differ from Markov processes in that the waiting time in a state need not be exponential. For these processes, if the mean waiting time m_j in every state j is finite, it is shown that

$$\Pr\{X_t=j | X_0\} \to Q_j m_j/l_j,$$

where Q_j is the probability of entering j given X_0 and l_j is the mean recurrence time to j. The methods are similar to the renewal theory methods introduced by Feller [Trans. Amer. Math. Soc. **67** (1949), 98–119; MR **11**, 255] to obtain the corresponding result for Markov chains.

D. Blackwell (Berkeley, Calif.).

★ **Blanc-Lapierre, A. Considérations sur certains processus ponctuels et sur des fonctions aléatoires associées.** Colloque sur l'analyse statistique, Bruxelles, 1954, pp. 25–55. Georges Thone, Liège; Masson & Cie, Paris, 1955.

On considère un événement susceptible de se réaliser à un instant quelconque. On dit qu'on a défini un processus stochastique ponctuel (S) si l'on connaît la probabilité $f_n(t_1, \cdots, t_n)dt_1\cdots dt_n$ pour qu'il se produise un événement entre t_1 et t_1+dt_1, le suivant entre t_2 et t_2+dt_2, $\cdots$, le $n$ème entre t_n et t_n+dt_n, où n est un entier donné arbitraire. La connaissance de f_n permet de calculer diverses probabilités utiles: probabilité conditionnelle pour que, $\theta_1, \cdots, \theta_k$ correspondant à k événements successifs, les événements suivants aient lieu dans les intervalles $(\theta_{k+1}, \theta_{k+1}+d\theta_{k+1}), \cdots, (\theta_{k+h}, \theta_{k+h}+d\theta_{k+h})$ — probabilité pour qu'un intervalle donné contienne n événements.

Divers cas particuliers de processes (S) sont examinés: Processus stationnaires: f_n ne dépend que des t_j-t_i; processus de renouvellement: la variable aléatoire $t_{j+1}-t_j$ dépend de t_j, mais non de ce qui précède t_j; processus sans mémoire: le futur par rapport à un instant α appartenant ou non à (S) est indépendant du passé. Divers exemples sont ensuite donnés, qui peuvent avoir des applications physiques: Processus de Poisson, processus obtenus par effacement d'instants dans un processus de Poisson; processus des „points milieux," obtenus en remplaçant les points t_j du processus donné par les milieux des intervalles (t_j, t_{j+1}).

A un processus ponctuel on peut associer une fonction aléatoire $X(t)$. On dit que $X(t)$ est d'ordre n si $X(t)$ est connue lorsqu'on connaît les résultats d'une épreuve sur les n valeurs des t_j qui précèdent t. La plus simple des fonctions du premier ordre est la date $\theta(t)$ du „choc" immédiatement antérieur à t. C'est une fonction aléatoire de Markoff. Les propriétés de $\theta(t)$ sont étudiées dans le cas où (S) est un processus de Poisson stationnaire, ainsi que celles de la fonction aléatoire $t-\theta(t)$, et celles de diverses fonctions $R[t-\theta(t)]$.

A la notion de fonction du second ordre, on peut associer des problèmes spatiaux et non plus temporels: $t_1, t_2, \cdots, t_n, \cdots$ sont des points sur une droite, $t_j=\theta(t)$ est le point de la suite t_n immédiatement à gauche de t; $t_{j+1}=\xi(t)$ est à droite de t. Quelles sont les propriétés de la liaison stochastique entre $\theta(t)$ et $\xi(t)$? L'étude est faite dans le cas où (S) est un processus de Poisson. On calcule en particulier, dans le cas stationnaire, la fonction de corrélation de la distance algébrique de t au milieu de l'intervalle (t_j, t_{j+1}).

Quelques exemples de fonctions d'ordre infini, représentées par

$$X(t)=\sum_{t_j<t} R(t-t_j)$$

sont enfin examinés, dans le cas où (S) est un processus de Poisson.

J. Bass (Paris).

See also: Halberstam, p. 461; Pinsker, p. 514; Hinčin, p. 567.

Mathematical Statistics

Bennett, B. M. Note on the moments of the logarithmic non-central χ^2 and z distributions. Ann. Inst. Statist. Math., Tokyo **7** (1955), 57–61.

The moments referred to in the title are given in terms of infinite series involving the digamma and its derivations.

I. R. Savage (Stanford, Calif.).

Chu, John T., and Hotelling, Harold. The moments of the sample median. Ann. Math. Statist. **26** (1955), 593–606.

The authors show that under certain regularity conditions, the central moments of the sample median are asymptotically equal to the corresponding moments of the asymptotic distribution, which is normal. A general approximation procedure involving expanding the inverse function of the c.d.f. in a Taylor series is given for the moments of the sample median. The approximation error can be made arbitrarily small by using a sufficiently large number of terms in the expansion. The method is applied to the normal, Laplace, and Cauchy distributions. An alternative and much simpler procedure gives upper and lower bounds for the variance of sample medians drawn from normal and Laplace distributions.

Of considerable practical importance are the detailed results concerning the variance of sample medians drawn from a normal parent distribution. A very useful result is the following. Let $\tilde{x}$ and $\tilde{\mu}_2$ be respectively the sample median and its variance of a sample of size $2n+1$ drawn from a normal distribution with unit variance. If $\bar{\mu}_2=\pi/2(2n+1)$ is the variance of the asymptotic distribution of $\tilde{x}$, then

$$B_n\left(1-\frac{1}{2n+2}\right)^{3/2} \leq \tilde{\mu}_2/\bar{\mu}_2 \leq B_n\left(1+\frac{1}{2n}\right)^{3/2},$$

where for all practical purposes and $n \geq 4$, $B_n \sim 1+(8n)^{-1}$. A more precise statement is to be found in Theorem 4 of the paper. *B. Epstein* (Stanford, Calif.).

Seal, K. C. A note on moments of order statistics from symmetrical populations. Calcutta Statist. Assoc. Bull. **6** (1955), 91–92.

Consider n populations having p.d.f. $f(x; \theta_i)$ $(i=1, 2, \cdots, n)$, each of which is symmetrical about the location parameter θ_i. Let a sample of size n be obtained by taking one observation from each population. Let x_i denote the observation taken from the ith population. Suppose that the θ_i's are chosen symmetrically about zero and consider the n order statistics $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ obtained by reordering the x_i's according to size. Then the author states various identities involving moments of order statistics which are symmetrically placed about the sample median. The results are immediate consequences of the double symmetry assumptions. *B. Epstein.*

Hodges, J. L., Jr. On the noncentral beta-distribution. Ann. Math. Statist. **26** (1955), 648–653.

In computing the noncentral beta-distribution function by the method of Nicholson [same Ann. **25** (1954), 607–610; MR **16**, 272] one must evaluate

$$P_j = \sum_{k=0}^{b-j-1} \left[(-1)^k \binom{b-j-1}{k} \frac{(a+b-1)(a+b-2)\cdots(a+j)}{(b-j-1)!(a+j+k)} \right] (1-x)^k,$$

which the author simplifies to

$$P_j = \sum_{t=0}^{b-j-1} \binom{A+t}{t} x^t, \quad A = a+j-1.$$

A table of $\binom{A+t}{t}$ to 7 significant figures is provided for $A = .5(1)19.5$, $t = 1(1)18$, adequate for computing all of the P_j when $2b \leq 40$, $2a+2b \leq 43$, and for the initial P_j for larger values of $2a$. This method is compared with that of Tang [Statist. Res. Mem. London **2** (1938), 126–149]. *I. R. Savage* (Stanford, Calif.).

Watson, G. S. The distribution of the ratio of two quadratic forms. Austral. J. Phys. **8** (1955), 402–407.

Two expressions are derived for the exact cdf (cumulative distribution function) of a type of ratios of weighted sums of independent chi-squares. This type includes certain serial correlation coefficients and for odd N the ratio

$$\sum_{i=1}^{N} (x_i - 2x_{i+1} + x_{i+2})^2 / \sum_{i=1}^{N} (x_i - x_{i+1})^2,$$

where $x_1, \cdots, x_N$ are normally, independently, and identically distributed, and $x_{N+1} = x_1$, $x_{N+2} = x_2$. Dixon [Ann. Math. Statist. **15** (1944), 119–144; MR **6**, 6] found the mean and variance of this ratio. The simpler of the expressions for the cdf is obtained by methods used by Box [ibid. **25** (1954), 290–302; MR **15**, 884]. *D. M. Sandelius* (Göteborg).

Rios, Sixto. Problems of maxima and minima related to inference in finite populations. Trabajos Estadist. **6** (1955), 3–30. (Spanish. English summary)

For a finite population $(X_1, \cdots, X_N)$, the author solves several maximum problems relating the mean μ, the variance σ^2, $\max x_i = M$, $\min x_i = m$, the range $r = M - m$, and the third moment α_3 about the mean. The problems solved are: $\max \sigma^2$ for fixed μ, m, M; $\max M$ for fixed μ, σ^2; $\max R$ for fixed μ, σ^2; $\max \sigma^2$ for fixed R; and $\max \alpha^3$ for fixed σ^2. *D. Blackwell* (Berkeley, Calif.).

Kamat, A. R. Modified mean square successive difference with an exact distribution. Sankhyā **15** (1955), 295–302.

The exact distribution is derived for a statistic δ'^2 based on a normal sample of $n=2m$ observations and obtained from the mean-square successive difference δ^2 by omission of the middle term. Corresponding F- and t-tests are derived. Upper 5% points of δ'^2 are given for $n = 8(2)12(4)24$. *D. M. Sandelius* (Göteborg).

Huzurbazar, V. S. Confidence intervals for the parameter of a distribution admitting a sufficient statistic when the range depends on the parameter. J. Roy. Statist. Soc. Ser. B. **17** (1955), 86–90.

The distributions are those whose densities are $g(x)/h(\theta)$ for $a(\theta) \leq x \leq b(\theta)$ and 0 otherwise, x and the parameter θ being real and a and b monotone in opposite directions. The sufficient statistic (based on independent $X_1, \cdots, X_n$ with common distribution function whose distribution is calculated is thus $\min [a^{-1}(\min_i X_i), b^{-1}(\max_i X_i)]$ if a is increasing. *J. Kiefer* (Ithaca, N.Y.).

David, H. A. A note on moving ranges. Biometrika **42** (1955), 512–515.

A series of n points, x_1 to x_n, plotted on a control chart in order of time are regarded as one sample and its range w_1 and maximum m_1 can be found. Moving on d points, the last n points x_{d+1} up to x_{d+n} define a new sample overlapping the first one. The correlation between the maxima m_1 and m_{1+d} and that between the ranges w_1 and w_{1+d} are derived in terms of order statistics and calculated numerically for normal distributions for $n=2$, 3, 4, and 5 and $d \leq n-1$.

In a series of N numbers, $N > n$, samples can be formed of all the possible sets of n consecutive numbers. The efficiency E of their mean range used as an "estimator" is tabulated for various N (10, 20, $\cdots$, 100, 200, ∞) and $n=2$ to 5, mainly for $d=1$. By this is meant an estimator for the standard deviation of a normal population, and the efficiencies are (rightly) given in terms of an estimate, based on the sample variance, that presumably has maximum (unit) efficiency. This is not pointed out in the table, and in the formula defining E the factor converting range to standard deviation has been left out. Finally the expected numbers of equal maxima and equal ranges are tabulated for various overlapping samples of n with $d=1$. *M. E. Wise* (Cardiff).

Bauer, R. K. Zur nichtparametrischen Ableitung der Streuungen des multiplen und des partiellen Korrelationskoeffizienten, sowie des multiplen Regressionskoeffizienten im Falle der Nullhypothese. Mitteilungsbl. Math. Statist. **7** (1955), 220–223.

Eckler, Albert Ross. Rotation sampling. Ann. Math. Statist. **26** (1955), 664–685.

The problem considered is the estimation of the time-dependent mean $\alpha(t_i)$ of a population. X_{ij} denotes the value of element j of the population at time t_i; it is assumed that the correlation between X_{ik} and X_{jk} is $\rho^{|i-j|}$ while the correlation between X_{ij} and X_{ik} is zero. When there is such a time correlation information contained in earlier samples may be used to improve the

current estimates of the population parameter, provided the two samples drawn at successive times have some elements in common. The name "rotation sampling" refers to the process of eliminating some of the old elements from the sample and adding new elements to the sample each time a new sample is drawn. In one-level rotation sampling the statistician can add to the sample pattern only sample values that have been drawn from the population at the current time; in two-level (and higher-level) rotation sampling, the statistician can add earlier sample values as well as current ones to the pattern. One-level rotation sampling estimates have been studied by several authors and in particular by Patterson [J. Roy Statist. Soc. Ser. B. **12** (1950), 241–255.] A fundamental theorem due to Patterson is one of the basic tools in the present paper where minimum-variance linear unbiased estimates are found for two-level and three-level rotational sampling. Comparison of the three methods of rotation sampling are made on a cost basis; also optimum patterns are determined for the one-level rotation sampling procedure. A number of related problems are also studied. *D. G. Chapman.*

Sen, A. R. On the selection of n primary sampling units from a stratum structure ($n \geqq 2$). Ann. Math. Statist. **26** (1955), 744–751.

"In this paper an outline is given of the general theory of the selection and estimation procedure for obtaining estimates of the between component of total error where first r primary sampling units are selected with probabilities proportional to some measure of size and the remaining $n-r$ are selected with equal probability, the selection being without replacement." [From the author's summary.] *J. Kiefer* (Ithaca, N.Y.).

Sarhan, A. E. Estimation of the mean and standard deviation by order statistics. III. Ann. Math. Statist. **26** (1955), 576–592.

[For parts I–II see same Ann. **25** (1954), 317–328; **26** (1955), 505–511; MR **15**, 809; **17**, 54.] The author finds best linear systematic estimates of the mean and standard deviation of certain populations from singly and doubly censored samples. A sample which is singly censored on the left (right) means that the smallest r_1 (largest r_2) observations are missing. A doubly censored sample is when the smallest r_1 and the largest r_2 observations are missing. Formulae for any sample size, n, and for any fixed r_1 and r_2 are given for the uniform and exponential distributions. The methods used are the same as in the case where there are no missing observations. Numerical results are given for several symmetric distributions for samples up to size 5. *B. Epstein* (Stanford, Calif.).

Chapman, Douglas G. Population estimation based on change of composition caused by a selective removal. Biometrika **42** (1955), 279–290.

This is a detailed study of a population-estimation method based on an initial sample, data on removal, and a second sample, and outlined in an earlier paper by the author [Ann. Math. Statist. **25** (1954), 1–15; MR **15**, 810]. The population is made up of two classes, and the removal is selective so that their relative sizes are changed. The author determines optimum allocation of the two samples and studies the effects on the population and class-size estimates of different "catchabilities" for the individuals in the two classes, and of bias and sampling errors in the removal data. The estimation method based on change of composition is compared with the one based on capture-recapture, and estimates based on a combination of both procedures are given. Further extensions include multiple sampling with several selective removals and the case when the variation between samples is larger than according to a binomial model. *D. M. Sandelius.*

Huntsberger, D. V. A generalization of a preliminary testing procedure for pooling data. Ann. Math. Statist. **26** (1955), 734–743.

The author considers estimates of θ_1 of the form

$$W(T)=\phi(T)\hat{\theta}_1+[1-\phi(T)](\sigma_2^2\hat{\theta}_1+\sigma_1^2\hat{\theta}_2)/(\sigma_1^2+\sigma_2^2)$$

where σ_1^2 and σ_2^2 are known and

a) θ_1 is $N(\theta_1, \sigma_1^2)$ and $\hat{\theta}_2$ is $N(\theta_2, \sigma_2^2)$,

b) $T=(\hat{\theta}_1-\hat{\theta}_2)/(\sigma_1^2+\sigma_2^2)^{\frac{1}{2}}$,

c) $0 \leqq \phi(T) \leqq 1$ and $\phi(T)=\phi(-T)$.

The bias and mean square error of $W(T)$ depend on $\gamma=(\theta_1-\theta_2)/(\sigma_1^2+\sigma_2^2)^{\frac{1}{2}}$. $W(T)$ is unbiased only when $\phi(T)=1$ and no choice of $\phi(T)$ uniformly minimizes the mean square error. The biases and mean square errors are found and compared for

i) $\phi(T)=T^2/(1+T^2)$,

ii) $\phi(T)=1$ if $T<t$ and $\phi(T)=0$ if $T \geqq t$,

iii) $\phi(T)=1-ae^{-bT^2}$, $0 \leqq a \leqq 1$, $b \geqq 0$.

Case ii) with $\sigma_1=\sigma_2$ was treated by Mosteller [J. Amer. Statist. Assoc. **43** (1948), 231–242], and Bennett [Ann. Inst. Statist. Math., Tokyo **4** (1952), 31–43; MR **14**, 665]. *I. R. Savage* (Stanford, Calif.).

Brunk, H. D. Maximum likelihood estimates of monotone parameters. Ann. Math. Statist. **26** (1955), 607–616.

Let $G(x, \theta)$ be the c.d.f. of a random variable and $\theta=\theta(t)$ an unknown real parameter depending on $t=(t^1, \cdots, t^n)$, an n-tuple of real numbers. The basic assumption is that $\theta(t) \leqq \theta(t')$ whenever every coordinate of t is less than or equal to every coordinate of t'. Suppose the experimenter chooses k t-vectors, $t_1, \cdots, t_k$ and makes observations on k independent random variables whose c.d.f.'s are $G(x, \theta(t_1)), \cdots, G(x, \theta(t_k))$. The problem is to make maximum likelihood estimates (m.l.e.) of $\theta_1=\theta(t_1), \cdots, \theta_k=\theta(t_k)$ in the presence of the monotonicity assumption on $\theta(t)$. The existence and consistency of the m.l.e. is established when G belongs to a certain "exponential" family which includes the binomial, normal, and Poisson. *M. Dwass* (Stanford, Calif.).

Ayer, Miriam, Brunk, H. D., Ewing, G. M., Reid, W. T., and Silverman, Edward. An empirical distribution function for sampling with incomplete information. Ann. Math. Statist. **26** (1955), 641–647.

Here is studied the problem of the paper reviewed above when $n=1$ and G is the distribution for a single binomial trial with success parameter $\theta=\theta(t)$. The m.l.e. of $\theta(t_1), \cdots, \theta(t_k)$ is given a useful characterization and its consistency is proved. The methods are somewhat different from those of the paper reviewed above.

M. Dwass (Stanford, Calif.).

Kraft, Charles. Some conditions for consistency and uniform consistency of statistical procedures. Univ. California Publ. Statist. **2** (1955), 125–141.

Conditions (which do not lend themselves to easy summary) that there be a consistent test or a uniformly consistent test for one set of distribution against another set of distributions, are given. Application is made to the likelihood-ratio test and to maximum-likelihood estimators. *J. Wolfowitz* (Ithaca, N.Y.).

van Eeden, Constance. A test against trend for a number of probabilities. Statistica Neerlandica **9** (1955), 131–159. (Dutch. English summary)

An expository account of two papers in English by the author and J. Hemelrijk [Proc. Nederl. Akad. Wetensch. Ser. A. **58** (1955), 191–198, 301–308; MR **17**, 56]. *H. A. David* (Melbourne).

Laderman, Jack. On the asymptotic behavior of decision procedures. Ann. Math. Statist. **26** (1955), 551–575.

For the problem of deciding among a finite number of distribution functions (d.f.'s) on the basis of n observations, the author defines, in the natural way, the "asymptotic minimaxity" and the "asymptotic admissibility" of a sequence $\{\delta_n\}$ of decision functions, and gives a few consequences of these definitions. The principal results are several concerned with the problem of deciding among k univariate normal d.f.'s with common known variance. A typical one is the following: Let σ^2 be the common variance, let $\theta_1<\theta_2<\cdots<\theta_k$ be the k means, and let $\gamma=\min_i(\theta_{i+1}-\theta_i)/\sigma$. If there is an index j such that $\theta_j-\theta_{j-1}>\gamma\sigma$, $\theta_{j+1}-\theta_j>\gamma\sigma$, then the sequence of minimax decision functions is asymptotically inadmissible. *J. Wolfowitz* (Ithaca, N.Y.).

Matusita, Kameo. Decision rules, based on the distance for problems of fit, two samples, and estimation. Ann. Math. Statist. **26** (1955), 631–640.

Let $\Omega=\{F\}$ be the totality of all distribution functions (d.f.'s) which assign probability one to k fixed points. The distance $\|F_1-F_2\|$ between any two elements F_1 and F_2 of Ω is defined as $\sum_{i=1}^{k}(\sqrt{\varrho_{1i}}-\sqrt{\varrho_{2i}})^2$, where ϱ_{ji} is the probability of the ith point according to F_j ($j=1, 2$; $i=1, \cdots, k$). We cite the following results as representative of the paper: 1) Let $X_1, \cdots, X_n$ be n independent identically distributed chance variables with the d.f. $F_0\in\Omega$, and let S_n be their empiric d.f. The author proves that, for any $t>0$,

$$P\left\{\|F_0-S_n\|^2<\frac{k-1}{n}t\right\}\geqq 1-\frac{1}{t}$$

and that, if $\|F-F_0\|^2\geqq 4(k-1)t/n$,

$$P\left\{\|F-S_n\|^2\geqq\frac{(k-1)}{n}t\right\}\geqq 1-\frac{1}{t}.$$

This leads to a test of goodness of fit: 2) Let $X_1, \cdots, X_n$, $Y_1, \cdots, Y_m$ be independently distributed, let $F\in\Omega$ be the common d.f. of the X's, and let $G\in\Omega$ be the common d.f. of the Y's. Let S_n be the empiric d.f. of $X_1, \cdots, X_n$, and let S_m' be the empiric d.f. of $Y_1, \cdots, Y_m$. The author proves that, when $F=G$,

$$P\{\|S_n-S_m'\|<\eta\}\geqq 1-\frac{4(k-1)}{\eta^2}\left(\frac{1}{n}+\frac{1}{m}\right)+\frac{16(k-1)^2}{\eta^4 nm}$$

and, when $\|F-G\|\geqq\delta$,

$$P\{\|S_n-S_m'\|\geqq\eta\}\geqq 1-\frac{4(k-1)}{(\delta-\eta)^2}\left(\frac{1}{n}+\frac{1}{m}\right)+\frac{16(k-1)^2}{(\delta-\eta)^4 nm}.$$

This leads to a test whether two samples are from the same population.

Other results in the paper are on estimation and on asymptotic distributions. *J. Wolfowitz.*

★ **Darmois, Georges. Sur la régression. Résultats nouveaux. Problèmes non résolus.** Colloque sur l'analyse statistique, Bruxelles, 1954, pp. 9–23. Georges Thone, Liège; Masson & Cie, Paris, 1955.

Statements of a variety of results and proofs of some. Typical considerations are (under various hypotheses) the implications of

(1) $F_1(x|y)=F_1(x-\alpha y-\beta|0)$ and

$$F_2(y|x)=F_2(y-\gamma x-\delta|0)$$

($F_1(\cdot|y)$ is the conditional d.f. of X given $Y=y$, etc.),

(2) $$\int y dF_2(y|x)=ax+b,$$

(3) $$X=S_0+S_1,\quad Y=S_0+S_2$$

with S_0, S_1, S_2 independent, with generalizations to more variables, etc. Some of the results have appeared in papers of S. Bernstein, C. R. Rao, S. F. Kenney, and many others. *J. Kiefer* (Ithaca, N.Y.).

Mine, Akiko. Estimation of linear regression coefficients in time series. (A note on the generalization of minimum distance method.) Ann. Inst. Statist. Math., Tokyo **6** (1955), 181–189.

The author estimates, by the minimum-distance method, the regression coefficients of a chance variable on given constants. Under certain conditions on the system of constants the estimators are shown to be strongly consistent. *J. Wolfowitz* (Ithaca, N.Y.).

Jowett, G. H. Least squares regression analysis for trend-reduced time series. J. Roy. Statist. Soc. Ser. B. **17** (1955), 91–104.

The treatment refers to situations where the shorter-term relations between the time-series are of relevance whereas the longer-term relations are disturbed by nuisance trends. Parameter estimates and standard errors being obtained by the use of his "local statistics" [Biometrika **42** (1955), 160–169; MR **16**, 1134], the author develops the approach by reducing the trend by a moving average operation and correcting for residual autocorrelation on the basis of a first-order Markov scheme. He stresses that even under these special circumstances the hypothesis testing will largely be subjective or depend upon ancillary information on the residual variation; this type of difficulty however is not due to the method but seems to be inevitable if the trend factors have to be excluded from the analysis. *H. Wold* (Uppsala).

Scott, J. F., and Small, V. J. A numerical investigation of least squares regression involving trend-reduced Markoff series. J. Roy. Statist. Soc. Ser. B. **17** (1955), 105–114.

For use in Jowett's approach [see the preceding review] the authors present auxiliary tables that cover the special case when the regression contains one explanatory variable, the trend is reduced by a moving average operation, and the residual follows a Markov scheme. *H. Wold.*

★**Bartlett, M. S. The statistical analysis of stochastic processes.** Colloque sur l'analyse statistique, Bruxelles, 1954, pp. 113–132. Georges Thone, Liège; Masson & Cie, Paris, 1955.

A review of some particular processes (mostly either stationary or of simple Markov type) and of methods proposed for their analysis. The final section indicates some general directions for further research.

P. Whittle (Wellington).

Sakaguchi, Minoru. On a problem of two populations. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **3** (1954), 65–69.

The paper represents a slight extension of a particular problem discussed in detail by H. Robbins [Bull. Amer. Math. Soc. **58** (1952), 527–535; MR **14**, 300]. Essentially the problem when stated in terms of "tossing" coins is the following: Given two coins (the probabilities of "heads" are unknown) the problem is to find a procedure (among a class of restricted procedures) which has the maximum expected number of "heads" in n tosses where before each toss the experimenter must select one of the two given coins and where the selection process is only allowed to depend on the result of the most recent toss.

M. Muller (Ithaca, N.Y.).

Thompson, W. A., Jr. On the ratio of variances in the mixed incomplete block model. Ann. Math. Statist. **26** (1955), 721–733.

In the variance component analysis of incomplete block designs two kinds of variances appear, i.e. the error variance within blocks and the variance of block effects. A method proposed by Wald for general regression problems gives exact confidence limits and tests of significance for the ratio of these variances. The present applies Wald's results to certain classes of designs and indicates the simplifications which occur in special cases. The efficiency of Wald's method relative to alternative procedures is not discussed. *P. Meier* (Baltimore, Md.).

Bay, Z., Henri, V. P., and Kanner, H. Statistical theory of delayed-coincidence experiments. Phys. Rev. (2) **100** (1955), 1197–1208.

See also: Binet, Leslie, Weiner and Anderson, p. 449; Méric, p. 500; Anderson, p. 536.

Theory of Games, Mathematical Economics

Franckx, Ed. Sur les jeux stratégiques finis. Colloque sur l'analyse statistique, Bruxelles, 1954, pp. 143–158. Georges Thone, Liège; Masson & Cie, Paris, 1955.

Let X be the convex hull of a finite set $(A_1, \cdots, A_m)$ of points in n-space, let Y be the convex hull of the n unit vectors $E_1, \cdots, E_n$ and let ϕ be a function on $X \times Y$ which is for fixed y (x) convex (concave) in x (y). Associated with the two-person zero-sum game with strategy spaces X, Y and payoff ϕ, is another such game (X^*, Y, ϕ^*), with X^* the convex hull of

$$A^*_1=(\phi(A_1, E_1), \cdots, \phi(A_1, E_n)), \cdots,$$

$$A^*_m=(\phi(A_m, E_1), \cdots, \phi(A_m, E_n))$$

and $\phi^*(x^*, y)=(x^*, y)$. It is shown that if x_0, y_0 is a solution of the original game, where $x_0=\sum_1^n \lambda_i A_i$, $\lambda_i \geqq 0$, $\sum \lambda_i=1$, then (x^*, y_0) is a solution of the new game, where $x^*=\sum \lambda_i A_i^*$, and that the two games have the same value. The convexity-concavity requirements on ϕ can be weakened if either game has a pure-strategy solution. An extension to n-person constant-sum games, replacing solution by equilibrium point (Nash), is given.

D. Blackwell (Berkeley, Calif.).

Wolfe, Philip. The strict determinateness of certain infinite games. Pacific J. Math. **5** (1955), 841–847.

The author settles a question on infinite games with perfect information raised by the reviewer and F. M. Stewart [Contributions to the theory of games, v. 2, Princeton, 1953, pp. 245–266; MR **14**, 999]. He shows that a win-lose game in which the winning set for player I is a G_δ in the natural topology of an infinite game, is strictly determined.

D. Gale (Providence, R.I.).

Nikaidô, Hukukane, and Isoda, Kazuo. Note on non-cooperative convex games. Pacific J. Math. **5** (1955), 807–815.

The authors generalize the Nash equilibrium-point theorem for n-person games [Ann. of Math. (2) **54** (1951), 286–295; MR **13**, 261] to convex games, that is games satisfying the conditions: (a) the ith player's strategy space is a compact convex subset X_i of a topological linear space E_i. (b) The ith player's pay off $K(x_1, \cdots, x_i, \cdots, x_n)$ is concave with respect to his own strategy variable $x_i \in X_i$. (c) The sum of the payoffs

$$\sum_{i=1}^{n} K_i(X_1, \cdots, X_n)$$

is continuous in the n variables $x_1, \cdots, x_n$. (d) For each fixed x_i, K_i is continuous simultaneously in the variables $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. A very simple proof is given using only the Brouwer fixed-point theorem. This is done by transforming the problem into an equivalent one by a generalization of the von Neumann symmetrization method.

D. Gale (Providence, R.I.).

Shubik, Martin. The uses of game theory in management science. Management Sci. **2** (1955), 40–54.

★**Samuelson, Paul A. Linear programming and economic theory.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 251–272. National Bureau of Standards, Washington, D.C., 1955.

A discursive expository paper touching on the sources of inspiration of linear programming in economic theory, the existence of competitive equilibrium, and the superiority for some purposes of convex-function methods to those of the calculus.

K. J. Arrow (Stanford, Calif.).

Danskin, J. M. Mathematical treatment of a stockpiling problem. Naval Res. Logist. Quart. **2** (1955), 99–109.

The problem is to maximize the expected utility of a stockpile designed to meet a future uncertain contingency. It is assumed that the probability $V(t)$ the contingency will occur before time t is known, and that $V(t)$ is absolutely continuous with density $v(t)$. The mathematical formulation is to maximize, by choice of $y(t)$, the functional

$$\Gamma(y)=\int_0^\infty F[x(t)]v(t)dt,$$

where $x(t)$ represents the stockpile size and $y(t)$ rate of

expenditure, subject to

$$x(t)=\bar{x}+\int_0^t H[y(\tau)e^{\alpha\tau}]d\tau,\ \bar{x}\geqq 0,\ \int_0^\infty y(t)dt=C,\ y(t)\geqq 0.$$

$\bar{x}$ represents the initial stockpile, α the interest rate, and C the total budget. The problem is solved under the following assumptions: (1) F and H are differentiable, strictly concave, and strictly increasing; (2) $F'(0)$ and $H'(0)$ are finite; (3) H is bounded; (4) $V(\infty)=1$, $v(t)$ is positive and decreasing; (5) there exists a constant $\beta>\alpha$ such that $v(t)\geqq\beta(1-V(t))$ for all $t\geqq 0$. The solution $y_0(t)$ is positive and strictly decreasing in some interval $(0, t_0)$, and $y_0(t)=0$ for $t>t_0$. $y_0(t)$ and t_0 are determined by a rather complicated pair of functional equations.

W. H. Fleming (Lafayette, Ind.).

Charnes, A., and Cooper, W. W. Optimal estimation of executive compensation by linear programming. Management Sci. 1 (1955), 138–151.

The authors show how linear programming may be used to arrive at a linear formula for estimating executive salaries so that (1) company rank is reflected by corresponding numerical rank of salary, (2) floor and ceiling requirements are satisfied, (3) the sum of the absolute values of deviations from current practices is minimized. They also indicate how the linear-programming point of view may be useful in other Chebyshev approximation problems, both computationally and theoretically.

A. J. Hoffman (Washington, D.C.).

★**Charnes, A., Cooper, W. W., and Mellon, B. A model for optimizing production by reference to cost surrogates.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 117–150. National Bureau of Standards, Washington, D.C., 1955.

A problem often arises in applying the abstract econometric concepts in the theory of the firm in that business and accounting records do not provide adequate information for optimizing decisions. Either cost information is almost wholly absent (small firms) or else the presence of multiple-product multiple-process conditions (large firms) makes the task of synthesizing such information a forbidding barrier to applying or testing of abstract models of business decision making.

Few business firms have available explicitly formulated cost or production functions with which to guide their operations. Nevertheless partial information or rules of thumb can be used to guide production decisions within certain limits. This paper shows how such particular information can be used as "surrogates" for total costs and made to yield a quantitative guide for production decisions which will yield minimum total costs in meeting sales schedules. The paper also prescribes the limits of such rules in applications to single product and multiple-product processes and suggests qualitative checks for determining whether these conditions are likely to be met in specific applications.

In general, the rule applies to any case involving arbitrary convex functions of cost. By cumulating sales requirements and determining the convex envelopes of the graphs, a method for determining optimum production schedules which meet sales requirements is developed which is simple and quickly applied when relatively few products and processes are involved. Observations are made on methods for effecting the necessary calculations to any desired degree of approximation in cases involving numerous commodities and processes. *T. L. Saaty.*

★**Salveson, M. E. The assembly line balancing problem.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 55–101. National Bureau of Standards, Washington, D.C., 1955.

The problem of balancing an assembly line is to distribute a set of partially ordered discrete work elements required in the production of a commodity over a set of assembly operatives in such manner as to satisfy the required production rate for the commodity and to minimize total idle or lost time of the operatives. It is similar to, but more general than the now classical "assignment problem" since it involves assigning an ordered set of tasks to each individual worker rather than a single task to each individual worker. Formal methods of solution together with "quick and dirty" practical methods of handling are discussed. The usefulness of the approach in practical factory situations also is demonstrated.

T. L. Saaty (Washington, D. C.).

Bellman, Richard. Equipment replacement policy. J. Soc. Indust. Appl. Math. 3 (1955), 133–136.

Hoffman, A. J., and Jacobs, Walter. Smooth patterns of production. Management Sci. 1 (1954), 86–91.

The following model for scheduling rates of production to meet a projected series of requirements is given together with a solution algorithm. Let the constants $r_i>0$ $(i=1, \cdots, n)$ be the shipping requirements for the various months. Let the variable $x_i\geqq 0$ $(i=1, \cdots, n)$ represent production in the various months. Let $R_t=\sum_{i=1}^{t} r_i$ and $X_t=\sum_{i=1}^{t} x_i$ $(t=1, \cdots, n)$. The shipping requirement is given as (1) $X_t\geqq R_t$ $(t=1, \cdots, n)$. Let S_t be the part of total production at the end of the tth month that has not been shipped and thus is stored. Thus, (2) $X_t-R_t=S_t$ $(t=1, \cdots, n)$. The problem is then to find $x_i\geqq 0$, $S_i\geqq 0$ $(i=1, \cdots, n)$ satisfying (2) which minimize the cost function (3) $\sum S_i+\lambda\sum y_t$, where $x_t=y_{t-1}-z_t$, $y_t, z_t, \lambda\geqq 0$ and $1/\lambda$ units of increased production cost the same as one unit of storage.

A principal result of this paper is the theorem: Let K_t be the convex envelope of the function R_t (with $R_0=0$). Then, for any prescribed value of λ, every solution $(X_1, \cdots, X_n)$ of the problem satisfies $R_t\leqq X_t\leqq K_t$. A simple solution is given in the case where the r_i are monotone increasing. *T. L. Saaty* (Washington, D.C.).

Antosiewicz, H., and Hoffman, A. A remark on the smoothing problem. Management Sci. 1 (1954), 92–95.

The formula providing a solution to the "smooth-patterns-of-production problem" [see the paper reviewed above] is extended to include the case where $0<r_1\leqq\cdots<r_{n-1}$; $r_{n-1}>r_n>0$. In fact, the formula holds if the r_i increase up to a certain value i_0, then decrease to $r_n>0$.

T. L. Saaty (Washington, D.C.).

Sherman, S. Comment on "Smooth patterns of production". Management Sci. 1 (1955), 271.

Without proof, the author points out that some of the conclusions of the paper reviewed second above continue to hold for a cost function $d(x_{i+1}-x_i)$ which is zero for non-negative $x_{i+1}-x_i$ and is a non-negative monotone increasing function of $x_{i+1}-x_i$ for non-negative $x_{i+1}-x_i$. However, Theorem 4 of that paper, which provides an algorithm for the optimum production schedule when the

delivery requirements increase from month to month, does not hold for some examples in which $d(x_{i+1}-x_i)=0$ for $x_{i+1}-x_i \leqq 0$ and is convex non-increasing for $x_{i+1}-x_i > 0$. *T. L. Saaty* (Washington, D.C.).

de Figueroa, Emilio. Leontief's system and the principle of the maximum. Rev. Ci. Apl. **9** (1955), 112–121. (Spanish)

It is well-known that if the production functions in a Leontief system are all of Cobb-Douglas type: $Y_i = A_i \prod_j x_{ij}^{b_{ij}}$ then competitive profit-maximization with given prices P_i leads to $b_{ij}=P_j x_{ij}/P_i Y_i$. Hence the constant output-elasticities are the observed Leontief input-coefficients as they are ordinarily measured [L. Klein, Rev. Econ. Studies **20** (1952), 131–136].

With the same assumption the author then considers the consequences of aggregating some of the sectors into an artificial sector. The aggregated variables are appropriately defined as value-weighted geometric means. It is shown that the condition of profit-maximization for the individual sectors is compatible with that for the aggregated sector. *R. Solow* (Cambridge, Mass.).

★ **McKenzie, Lionel W. Competitive equilibrium with dependent consumer preferences.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 277–294. National Bureau of Standards, Washington, D.C., 1955.

The technology is assumed to satisfy the usual conditions of linear programming, with the additional condition that the set of attainable commodity points must have an interior point. For each individual i, there is a set W^i, closed, convex, and bounded from below, which contains all combinations of commodities supplied and demanded which the individual can support. For each individual, there is a demand function which may depend not only on prices but also on the commodity combinations w^j of other individuals; the demand functions are assumed continuous. It is finally assumed that within a suitable range, the net expenditures of each consumer are zero. A competitive equilibrium, in this model, has the properties that each activity has a non-positive profit level, any activity actually carried on has a zero profit level, and supply equals demand for each commodity (note that prices may be negative and that disposal activities, if they exist, are included under demand). By the use of Kakutani's fixed-point theorem, the existence of a competitive equilibrium is demonstrated. The paper concludes with some remarks on consumer's behavior underlying the demand assumptions.

A number of misprints have been noted by the author, of which the following may cause trouble: (1) on the last line of footnote ** on p. 291, insert, "the subset of," before, "V"; (2) on p. 292, line 4, replace y^0 by v^0; (3) same page, line 6, replace, "maximal", by "<1"; (4) same page, line 15, replace the subscript i by k; (5) the formula in line 4 of p. 293 should read, "$v \bigcirc\!\!\!

morphisms is in that case isomorphic to the multiplicative group of p-adic integers (loc. cit., p. 241). It remains to be seen if these interesting methods can be extended to all abelian formal Lie groups and yield new proofs of the recent results of the reviewer [ibid. 77 (1955), 429–452; MR 17, 174] and perhaps new insight in the more difficult theory of non-abelian formal Lie groups. *J. Dieudonné.*

Rådström, Hans. Some elementary functional equations and Hilbert's fifth problem. Nordisk Mat. Tidskr. 3 (1955), 129–147, 182–183. (Swedish. English summary)

The author gives an elegant and elementary proof of the theorem that every 1-dimensional Euclidean local group is locally isomorphic to the real numbers under addition. As a preliminary, he discusses topological semigroups, with operation $x \circ y$, defined on an interval I of the real number system. If the operation is differentiable with derivative vanishing nowhere, then $x \circ y = f^{-1}(f(x)+f(y))$, where f is a strictly monotone differentiable function on I. He also discusses the functional equation $F(x \circ y) = F(x)+F(y)$, which plays an important rôle in his proof. He closes with some historical remarks on Hilbert's fifth problem, remarking that it is still open for Euclidean local groups. *E. Hewitt* (Princeton, N.J.).

Krafft, Günther. Die stetigen Darstellungen der reellen Formen der komplexen unimodularen, orthogonalen und symplektischen Gruppen. Mitt. Math. Sem. Giessen no. 53 (1955), i+53 pp.

The linear representations of the connected complex classical groups have all been determined long ago by E. Cartan, H. Weyl, I. Schur and R. Brauer. The author first remarks that this determines also immediately the linear representations (over the complex field) of any simply connected real form of the classical groups, by going over to its complexification. The bulk of his paper consists in examining how, from the so-called "fundamental" representations of a simply connected real form, one can obtain the linear representations of the nonconnected or non simply connected real forms of the classical groups; this essentially is reduced to the relations between representations of a group and of a normal subgroup of order 2, a question which has been extensively treated by H. Weyl and A. Clifford.

The author is apparently unaware of the literature on classical groups since 1940 [his bibliography does not list any item posterior to Weyl's "Classical groups", Princeton 1939; MR 1, 42]. He does not seem to know, for instance, how the real forms of the classical groups which he considers and which leave invariant several bilinear forms can be considered as classical groups over the sfield of quaternions; this would have greatly simplified his clumsy treatment of the determination of the connected components of these groups, which can be done easily, either by induction on the dimension as in Chevalley's "Lie groups", or by using generators of a simple type for the classical groups on quaternions, and applying the same kind of argument the author uses for classical groups over the real or complex field. *J. Dieudonné.*

Kostant, Bertram. On the conjugacy of real Cartan subalgebras. I. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 967–970.

A number of theorems are stated which provide the means for classifying the Cartan subalgebras of a real simple Lie algebra L_0 with respect to conjugacy under the full automorphism group, or under the adjoint group, A, of L_0. The classification procedure uses a fixed Cartan subalgebra, H, of the complexification of L_0, and the corresponding Weil group, W; in fact, the A-conjugacy classes of the Cartan subalgebras of L_0 are put in correspondance with the W-conjugacy classes of certain subspaces of H. The actual results cannot be profitably restated here, since they involve quite a variety of auxiliary notions. *G. P. Hochschild.*

Gurevič, G. B. Standard Lie algebras. Mat. Sb. N.S. 35(77) (1954), 437–460. (Russian)

All Lie algebras considered are linear Lie algebras, i.e. Lie algebras of linear transformations of a vector space $\mathfrak{B}$ of dimension n over a field of characteristic zero. Such a Lie algebra is called a nilalgebra, if all its elements are nilpotent. For each nilalgebra $\mathfrak{A}$ acting on $\mathfrak{B}$ there is a chain $0=\mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \cdots \subset \mathfrak{B}_m = \mathfrak{B}$ such that (1) $A\mathfrak{B}_k \subseteq \mathfrak{B}_{k-1}$ $(k=1, \cdots, m)$ for all $A \in \mathfrak{A}$. The set of all linear transformations A satisfying (1) is called a complete nilalgebra. If $\mathfrak{A}$ is any linear Lie algebra, then its normaliser $\mathfrak{S}$ is the set of all matrices S such that $[A, S] \in \mathfrak{A}$ for all $A \in \mathfrak{A}$. Now a standard Lie algebra is defined as a linear Lie algebra whose normaliser is also the normaliser of some complete nilalgebra. The author determines all standard Lie algebras by finding a basis for each such algebra in terms of the matrix units E_{ij} relative to a suitably chosen "canonical" basis of $\mathfrak{B}$: A standard nilalgebra is completely determined by its characteristic

$$\begin{pmatrix} i_1, & \cdots, & i_q \\ j_1, & \cdots, & j_q \end{pmatrix},$$

where $i_s \leq j_s$, $1 \leq i_1 < \cdots < i_q \leq n$, $1 \leq j_1 < \cdots < j_q \leq n$; its basis consists then of all E_{xy} such that $i_{s-1} < x \leq i_s$, $j_s < y$ $(s=1, \cdots, q;\ i_0=0)$. Next, if $\mathfrak{A}$ is any soluble standard algebra and $\mathfrak{S}$ its normaliser, then $[\mathfrak{A}, \mathfrak{S}]=\mathfrak{F}$ is a standard nilalgebra, and $\mathfrak{A}$ may be obtained from $\mathfrak{F}$ by the adjunction of certain diagonal matrices (in the canonical basis). Finally, an insoluble standard algebra consists of a number of square blocks on the main diagonal, together with a soluble standard algebra which turns out to be the radical of the whole. The blocks constitute a semi-simple algebra and thus a Levi-decomposition is obtained. Secondly the author investigates the embeddings of an arbitrary nilalgebra in a standard nilalgebra. Two linear transformations A, B are orthogonal, if $\mathrm{Tr}(AB)=0$. The orthogonal complement $\mathfrak{A}^\perp$ of a Lie algebra $\mathfrak{A}$ is again a Lie algebra. A nilalgebra $\mathfrak{H}$ is said to be minimally embedded in a standard nilalgebra $\mathfrak{A}$ if $\mathfrak{H} \subseteq \mathfrak{A}$, and if no proper subalgebra of $\mathfrak{A}$ containing $\mathfrak{H}$ is a standard algebra. It is shown that a nilalgebra $\mathfrak{H}$ is minimally embedded in a standard nilalgebra $\mathfrak{A}$ if and only if $\mathfrak{H}^\perp \cap [\mathfrak{S}^\perp, \mathfrak{A}]^\perp \subseteq \mathfrak{A}^\perp$, where $\mathfrak{S}$ is the normaliser of $\mathfrak{A}$. The proof uses the reduction of $\mathfrak{H}$ to a canonical form and hence leads, for given $\mathfrak{H}$, to a minimal embedding. As a corollary the author shows that a linear nilalgebra $\mathfrak{A}$ is a standard nilalgebra if and only if $[\mathfrak{A}, \mathfrak{A}^\perp]$ is a nilalgebra.

P. M. Cohn (Zbl 56, 265).

Dixmier, J. Sur les algèbres dérivées d'une algèbre de Lie. Proc. Cambridge Philos. Soc. 51 (1955), 541–544.

A theorem is proved on the derived algebras of a Lie algebra which is the analogue of a theorem of Philip Hall on p-groups. [Proc. London Math. Soc. (2) 36 (1933), 29–95, Th. 2.57]. The theorem asserts that the quotient algebra of the ith derived algebra modulo the $(i+1)$st derived algebra in a nilpotent Lie algebra is of dimension

at least 2^i+1 if the ith derived algebra is not abelian. The proof also parallels that of Philip Hall, using a chief series for a representation space for the algebra and an inductive argument. The numbers 2^i+1 are not always attainable. If a nilpotent algebra over a field of characteristic not two has dimension three for the quotient of its first derived algebra modulo its second, then he shows that the second is of dimension zero or one. As a sort of counter example to this counter example, he shows that the assumption of characteristic not two is essential to this last argument. *Marshall Hall, Jr.* (Columbus, Ohio).

See also: Chevalley, p. 457; Gel'fand and Fomin, p. 514; Kuiper and Yano, p. 528; Horie, p. 528.

Topological Vector Spaces

Iséki, Kiyoshi, and Kasahara, Shouro. Some properties of convex sets in linear spaces. Univ. Lisboa. Revista Fac. Ci. A. (2) **3** (1954–1955), 238–242.

The authors wish to supply elementary proofs of the separation theorem for convex bodies, and of a related result due to Kakutani [Proc. Imp. Acad. Tokyo **13** (1937), 93–94] (not listed in their bibliography). Proof of the latter is virtually identical with that of Kakutani. Proof of the former is based on a special case (not established in the paper) from which it can be quickly deduced by the method of Tukey [Portugal. Math. **3** (1942), 95–102; MR **4**, 13]. However, the authors follow a more devious path. Their assumption of local convexity in the separation theorem is unneeded and unused.

V. L. Klee (Los Angeles, Calif.).

Kasahara, Shouro. Note sur la topologie de l'espace $L(E, E)$. Math. Japon. **3** (1955), 93–96.

Kasahara, Shouro. Sur l'espace des endomorphismes continus de l'espace vectoriel localement convexe. Math. Japon. **3** (1955), 111–116.

Let E, F be locally convex topological linear spaces, $L(E, E)$ the algebra of continuous linear transformations from E into E with the topology of uniform convergence on some family of bounded subsets of E whose union is E, and similarly for $L(F, F)$. Let A be a subalgebra of $L(E, E)$ containing all transformations with finite-dimensional range, and for $B, N \subset E$, let $W(B, N)$ denote $\{u \in A \mid u(B) \subset N\}$. The following theorems are proved, with the final versions appearing in the second paper. Theorem 1: If there exist neighborhoods N, N' of 0 in E with $N' \neq E$, and bounded sets B, B' in E with $B' \neq \{0\}$, such that $[W(B, N)]^2 \subset W(B', N')$ (which is certainly true if multiplication in A is jointly continuous), then E is normable. Theorem 2: Any isomorphism between $L(E, E)$ and $L(F, F)$ is generated, in a canonical fashion, by an isomorphism between E and F. [Reviewer's note: Theorem 1 is also proved (though not all of it explicitly stated) by Blair, Proc. Amer. Math. Soc. **6** (1955), 209–210; MR **16**, 935. Theorem 2 was proved by Andô, Proc. Japan Acad. **30** (1954), 435–436; MR **16**, 373.] *E. Michael.*

Pavel, Monica. Théorèmes d'extension dans les espaces vectoriels normés. Com. Acad. R. P. Romîne **5** (1955), 1133–1138. (Romanian. Russian and French summaries)

Some elementary remarks, introducing the notions of linear retract and solid linear space. (The first half of the proof of Theorem 1 seems to be incorrect, but becomes trivial upon taking $F=E_1$ and f the identity map on E_1.)

V. L. Klee (Los Angeles, Calif.).

Marinescu, G. Complément à la note "Sur le prolongement des fonctionnelles linéaires continues dans les espaces vectoriels topologiques". Acad. R. P. Romîne. Bul. Şti. Secţ. Şci. Mat. Fiz. **7** (1955), 543–544. (Romanian. Russian and French summaries)

The author corrects an omission in an earlier paper, and indicates some additions to its bibliography [Marinescu, same Bul. **5** (1953), 317–326; MR **16**, 932].

V. L. Klee (Los Angeles, Calif.).

Tillmann, Heinz Günther. Dualität in der Funktionentheorie auf Riemannschen Flächen. J. Reine Angew. Math. **195** (1956), 76–101 (1955).

This paper is closely related to a paper by A. Grothendieck [same J. **192** (1953), 35–54; MR **15**, 438]. The essentially new element is introduced into this paper by the consideration of analytic functions defined on Riemann surfaces rather than on "schlicht" domains. Let $\mathfrak{M}$ be a point set on the Riemann surface $\mathfrak{R}$. Let E be a locally convex complex topological linear space (with Hausdorff separation), and let $\mathfrak{H}(\mathfrak{M}, E)$ be the linear space of all E-valued locally analytic functions on $\mathfrak{M}$. There is defined for $\mathfrak{H}(\mathfrak{M}, E)$ a topology which is closely associated with uniform convergence. The study of properties of $\mathfrak{H}(\mathfrak{M}, E)$ depends upon a suitable version of the Cauchy integral formula for functions on a Riemann surface; the work of Behnke and Stein [Math. Ann. **120** (1949), 430–461; MR **10**, 696] is used in this connection. The dual of $\mathfrak{H}(\mathfrak{M}, E)$ is investigated. Each continuous linear functional is associated with a uniquely determined function which is locally analytic on $\mathfrak{R}-\mathfrak{M}$ and has its value in the dual space E. The representation of the functional utilizes a contour integral. Finally, the author studies continuous linear mappings of $\mathfrak{H}(\mathfrak{M}, E)$ into a locally convex topological linear space F. Each such mapping is associated with a locally analytic function on $\mathfrak{R}-\mathfrak{M}$ to the space of continuous linear mappings of E into F. This function has a special property known as bi-analyticity, which is defined in terms of representability by an integral, corresponding to Cauchy's integral formula. If $\mathfrak{R}$ is the open complex plane and $\mathfrak{M}$ is bounded, bi-analyticity on $\mathfrak{R}-\mathfrak{M}$ involves being regular at ∞ and zero there.

A. E. Taylor (Los Angeles, Calif.).

Cristescu, Romulus. De la définition du produit en K-espaces. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 553–564. (Romanian. Russian and French summaries).

Reproduction in somewhat simplified form of the definition of product in a complete vector lattice due to Vulih [Mat. Sb. N.S. **22**(64) (1948), 27–78, 267–317; MR **10**, 46]. An extension is made to products of elements lying in two distinct vector lattices. *E. Hewitt.*

See also: Morse and Transue, p. 469; Nikaidô and Isoda, p. 506; Klee, p. 516; Ghika, p. 523.

Banach Spaces, Banach Algebras

Bartle, Robert G. On compactness in functional analysis. Trans. Amer. Math. Soc. **79** (1955), 35–57.

In this paper the author studies pairing of two sets, X

and Y, defined by a real-valued function h on $X \times Y$ such that each section of h is bounded. Using the notion of quasi-uniform convergence due to Arzelà [Rend. Accad. Sci. Ist. Bologna **1883–84**, 79–84] the author defines weak and strong pairing of sets X and Y in terms of quasi-uniform and uniform convergence of universal nets [Kelley, General topology, Van Nostrand, New York, 1955, p. 81; MR **16**, 1136] in X or in Y. Conditions are given that a subset X of a Banach space B [B^*] be paired by the natural function $h(x, x^*)=x^*(x)$ with the unit sphere of B^* [of B]. This gives a simple proof of the theorems of Schauder [Studia Math. **2** (1930), 183–196] and Gantmakher [Mat. Sb. N.S. **7(49)**, 301–308 (1940); MR **2**, 224] that a linear operator in a Banach space is compact (weakly) if and only if the adjoint operator is also compact (weakly). Other applications concern compactness in the important topologies in continuous function spaces. *M. M. Day* (Urbana, Ill.).

Kadec, M. I. On topological equivalence of uniformly convex spaces. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 137–141. (Russian)

The author proves that any two separable, uniformly convex Banach spaces [Clarkson, Trans. Amer. Math. Soc. **40** (1936), 396–414] are homeomorphic. To sketch the proof: Define in a uniformly convex B a sequence $P_0 \supset P_1 \supset \cdots \supset P_n \supset \cdots$ of closed linear subspaces so that the deficiency of P_n in B is n; then each P_n splits P_{n-1} into two half-spaces, labelled P_{n-1}^+ and P_{n-1}^-. For each x in B let x_n be the point of P_n nearest to x and define $H_n(x)=|x-x_n|$, $H_{n-1}(x)$, or $-|x-x_n|$ according as x_{n-1} is in P_{n-1}^+, P_n, or P_{n-1}^-. Theorem 1 asserts that a sequence $\{h_n\}$ determines a unique x in B such that $H_n(x)=h_n$ for all $n \geqq 1$ if and only, if, for all $j \geqq 1$, (1) $|h_j| \leqq |h_{j+1}|$ and (2) if $|h_j|=|h_{j+1}|$, then $h_j=h_{j+1}$: then $|x|=\lim_n |h_n|$. Applying this correspondence to two uniformly convex spaces gives the homeomorphism between them. *M. M. Day* (Urbana, Ill.).

Veloso, Maria Joana Bénard de Costa Sousa. La notion de dérivée en points frontières dans le cas des espaces de Banach. Univ. Lisboa. Revista Fac. Ci. A. (2) **3** (1954–1955), 255–278.

This paper complements and extends a paper by J. Sebastião e Silva [same Rev. (2) **1** (1950), 117–166; 401–402 (1951); MR **13**, 45]. If S and S^* are Banach spaces, and f is a function on S to S^*, the Fréchet differential of f at $x=a$, with argument h, is denoted by $f'(a)h$. Here $f'(a)$ is a member of $L(S, S^*)$, the space of bounded linear operators on S to S^*; $f'(a)$ is called the derivative of f at a. It is known that $f'(a)$ is unique at an interior point of the domain of definition M of f, provided f is differentiable at a. The author defines a point a on the boundary of M to be conically accessible if there exists an open set D contained in M such that the convex hull of $D \cup (a)$ lies in M except perhaps for a itself. Then it is shown that if f is differentiable at a conically accessible boundary point of M, the derivative there is unique. Various results given by Silva are then extended by showing that conically accessible boundary points can be admitted instead of dealing entirely with interior points. For example, if Γ is a rectifiable curve with end-points a, b, the formula

$$\int_\Gamma f(x)dx=f(b)-f(a)$$

is shown to hold, under suitable conditions on f, if Γ is composed of interior points and conically accessible boundary points of a convex set on which f is differentiable. *A. E. Taylor* (Los Angeles, Calif.).

Orlicz, W. On perfectly convergent series in certain functional spaces. Prace Mat. **1** (1955), 393–414. (Polish. Russian and English summaries)

Eine Reihe $\sum x_n$ (in einem B-Raum E) heißt vollkommen konvergent, wenn jede Teilreihe $\sum \eta_n x_n$ ($\eta_n=0$ oder 1) konvergiert. Bei unendlich-dimensionalem E folgt aus der vollkommenen Konvergenz nicht die absolute, wie für spezielle Räume leicht einzusehen ist und allgemein von Dvoretzky und Rogers [Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 192–197; MR **11**, 525] bewiesen wurde. Verfasser gibt Kriterien folgender Art: Konvergiert $\sum \eta_n x_n$ oder $\sum \eta_n \xi(x_n)$ für genügend viele η_n und genügend viele stetige Linearformen ξ, so ist $\sum x_n$ vollkommen konvergent. Im Raum M der beschränkten meßbaren Funktionen ist vollkommene Konvergenz gleichbedeutend mit der gleichmäßigen Konvergenz (bis auf eine Nullmenge) von $\sum |x_n(t)|$. Es folgen Anwendungen, die die Konstruktion stetiger oder nichtdifferenzierbarer Funktionen und die Theorie der Orthogonalreihen betreffen.

K. Zeller (Tübingen).

Orlicz, W. Linear operations in Saks spaces. II. Studia Math. **15** (1955), 1–25.

[For definitions and such notations as are used in this review the reader is referred to Orlicz, Studia Math. **11** (1950), 237–272; MR **12**, 418.] The study of Saks spaces finds its motivation in the renorming of the unit sphere of a Banach or Fréchet space with a norm relative to which the unit sphere is complete. Such renormings arise naturally; e.g., in the space X of bounded sequences with sup norm a new norm $\|\{a_n\}\|^*$ defined by $\sum_1^\infty 2^{-n}|a_n|$ can be introduced. So renormed the unit sphere of X becomes the Saks space X_s. In the present paper, the author studies sequences of linear operations U_n from X_s into a Banach or Fréchet space Y. Associated with the sequence U_n is $w(x)$, a measure of the oscillation of the sequence U_n at x and sets R_σ, S_σ where $w(x) \geqq \sigma$, resp. $<\sigma$. Various special values and limits of σ and $w(x)$ are related to the behavior of the sequence U_n. A collection of category theorems reminiscent of the Banach-Steinhaus "Ideenkreis" is derived. Applications are given to the theory of summability of the discrete and continuous types. Details are numerous and extensive and only the barest bones of the paper can be communicated through a review.

B. R. Gelbaum (Minneapolis, Minn.).

Albrycht, J. Theorems of Saks' type for abstract polynomials. Studia Math. **15** (1955), 26–28.

A result of Alexiewicz [Studia Math. **14** (1954), 1–12; MR **15**, 963] is generalized through the use of an earlier theorem of the author [ibid. **14** (1954), 79–81; MR **15**, 963]. Notations and definitions can be found in the cited reviews. Theorem 1. If the set B is linear and analytic, then there exists a decomposition $T=e \cup h$ and a residual set R in X such that (a) for every x and every $\epsilon>0$ there exists a set e' such that $\mu(e \setminus e')<\epsilon$ and $U(x)_{e'} \epsilon B$; (b) for every $x \epsilon R$ and every set $h' \subset h$ of positive measure $U(x)_{h'} \notin B$. Theorem 2. When the set B is analytic and satisfies the following condition: from $y_{e_n} \epsilon B$ ($n=1, 2, \cdots$) and $e=\mathbf{U}_{n=1}^\infty e_n$ results $y_e \epsilon B$, then there exists a decomposition $T=e \cup h$ and a residual set $R \subset X$ such that (a_1) $U(x)_e \epsilon B$ for every x, (a_2) $U(x)_{h'} \notin B$ for every $x \epsilon R$ and every set $h' \subset h$ of positive measure.

B. R. Gelbaum (Minneapolis, Minn.).

Kopeć, J., and Musielak, J. On the estimation of the norm of the n-linear symmetric operation. Studia Math. **15** (1955), 29–30.

If X and Y are normed linear spaces a continuous symmetric n-linear on X^n to Y is a function $U(x_1, \cdots, x_n)$ with values in Y which is linear as a function of x_k $(k=1, \cdots, n)$ and unaltered in value by a permutation of $x_1, \cdots, x_n$. If $x_1=\cdots=x_n=x$, $U(x_1, \cdots, x_n)$ is denoted by $U(x)$ and called a power of degree n. Let

$$M_n = \sup \|U(x_1, \cdots, x_n)\| \text{ for } \|x_k\| \leqq 1,$$

$$P_n = \sup \|U(x)\| \text{ for } \|x\| \leqq 1.$$

The present paper shows by an example that for each n it is possible to have $M_n=(n^n/n!)P_n$. The spaces chosen are $X=L(0, 1)$, Y the space of reals. It has long been known that $M_n \leqq (n^n/n!)P_n$. This result is given in the thesis of R. S. Martin [California Inst. Tech. 1932; unpublished], and the argument is reproduced in a paper by the reviewer [Tôhoku Math. J. **44** (1938), 302–318]. It was proved by Banach [Studia Math. **7** (1938), 36–44] that $M_n=P_n$ if X is a separable Hilbert space and Y is any normed linear space. The reviewer, in a talk at Princeton in 1938, gave an example in which $M_n=(n^n/n!)P_n$. It is $U(x)=\xi_1\xi_2\cdots\xi_n$ with X the n-dimensional space of $x=(\xi_1, \cdots, \xi_n)$ with $\|x\|=|\xi_1|+\cdots+|\xi_n|$ and Y the space of scalars. *A. E. Taylor* (Los Angeles, Calif.).

Bonsall, F. F. Indefinitely isometric linear operators in a reflexive Banach space. Quart. J. Math. Oxford Ser. (2) **6** (1955), 179–187.

Let V be a reflexive real Banach space, ϕ a non-zero continuous linear functional on V, e a point of V such that $\phi(e)=1$, P the projection given by $Px=\phi(x)e$ $(x \in V)$, and Q the orthogonal projection $I-P$. For a positive real t, let $q(x)=\|Qx\|^t-|\phi(x)|^t$ $(x \in V)$. The author studies linear operators which are "bounded" or "isometric" relative to the "indefinite norm" q, justification for the work being that rather precise information on their spectra is obtained (the spectrum being defined as that of the natural complex extension of the operator). (An example of the situation studied has $V=(l^p)$ with $1<p<\infty$, and $q(x)=-|x_1|^p+\sum_2^\infty |x_i|^p$ for $x=(x_1, x_2, \cdots) \in (l^p)$.) The principal result asserts (in part) that if V, Q, and q are as above, T is a linear operator on V such that $qT=q$, and some slight additional hypotheses are satisfied, then T and T^{-1} are bounded linear operators with spectral radii ϱ and σ respectively such that $\sigma^{-1} \leqq 1 \leqq \varrho$; the numbers ϱ and σ^{-1} are characteristic numbers of T, and if either differs from 1 it is a simple characteristic value; the rest of T's spectrum is contained in the unit circumference. The results are quite similar to those obtained by Krein and Rutman [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 1(23), 3–95; MR **10**, 256] for Lorentz mappings in (l^2).

V. L. Klee (Los Angeles, Calif.).

Šilov, G. E. On the decomposition of a commutative normed ring into a direct sum of ideals. Amer. Math. Soc. Transl. (2) **1** (1955), 37–48.

Translated from Mat. Sb. N.S. **32(74)** (1953), 353–364; MR **14**, 884.

Warner, Seth. Polynomial completeness in locally multiplicatively-convex algebras. Duke Math. J. **23** (1956), 1–11.

Dans la plus grande partie de ce travail, l'auteur considère des algèbres séparées (sur le corps réel ou le corps complexe) multiplicativement convexes, c'est-à-dire dans lesquelles il y a un système fondamental de voisinages convexes de 0 tels que $V^2 \subset V$. Supposons pour simplifier qu'il s'agisse d'algèbres E ayant un élément unité (l'auteur ne fait pas cette hypothèse, et introduit partout les „adverses" à la place des inverses); l'auteur considère alors la propriété suivante, plus faible que le fait pour E d'être complète: (A) tout filtre de Cauchy $\mathcal{F}$ pour lequel il existe $a \in E$ tel que ax et xa tendent vers 1 suivant $\mathcal{F}$ est convergent dans E. Il prouve qu'il revient au même, dans la représentation usuelle de E comme sous-algèbre d'un produit d'algèbres de Banach E_V, de supposer que tout élément de E, dont toutes les projections sont inversibles, est lui-même inversible. Par suite, pour ces algèbres, la théorie spectrale développée par E. Michael pour les algèbres multiplicativement convexes et complètes [Mem. Amer. Math. Soc. no. **11** (1952); MR **14**, 482] s'étend sans changement. Pour une algèbre normée, la propriété (A) équivaut au fait que $1-x$ est inversible pour $\|x\|<1$; il y a des relations analogues dans le cas général entre ces algèbres et les Q-algèbres de Kaplansky (algèbres où les éléments inversibles forment un voisinage de 1). Enfin l'auteur généralise les définitions précédentes à des anneaux topologiques quelconques et montre (généralisant un résultat de Kaplansky) que dans un anneau possédant la propriété (A) et tel que $x \to x^{-1}$ soit uniformément continu, l'ensemble des éléments inversibles est fermé et discret ou parfait, et que le radical est fermé.

J. Dieudonné (Evanston, Ill.).

Wermer, John. Maximal subalgebras of group-algebras. Proc. Amer. Math. Soc. **6** (1955), 692–694.

Let G be an ordered, abelian, discrete group; G^+, the non-negative elements of G; L, the group algebra of G; L^+, the closed subalgebra of L consisting of all functions that vanish outside of G^+. Theorem: L^+ is maximal (i.e., is not contained in any larger proper closed subalgebra) if and only if the ordering of G is archimedian. The theorem has also been proved by Arens and Singer [to be published in Trans. Amer. Math. Soc.]. *M. Jerison.*

Goldman, Malcolm. Structure of AW^*-algebras. I. Duke Math. J. **23** (1956), 23–34.

The author proves, by adopting Dixmier's techniques, that a finite AW^* algebra A for which there exist sufficiently many positive linear maps to its center Z, actually has a (unique) central-valued trace. Such a trace, it is then shown, can be used to exhibit A as an AW^* subalgebra of an algebra B of type I. In case the spectrum of Z possesses a "perfect" measure, B will be a ring of operators, and A will have finite weak closure. *J. Feldman.*

Waelbroeck, Lucien. Le calcul symbolique dans les algèbres commutatives. C. R. Acad. Sci. Paris **238** (1954), 556–558.

Der von der Theorie der Banachalgebren bekannte symbolische Kalkül, der die Ausdrücke $f(a)$, a Element der Algebra, f eine analytische Funktion, für gewisse f erklärt, wird auf kommutative Algebren A mit Einselement, die zugleich vollständige lokalkonvexe komplexe Räume sind, übertragen. Es wird sogar $f(a_1, \cdots, a_n)$ für endlich viele $a_i \in A$ erklärt, f eine analytische Funktion von n Variablen, die auf einer als Spektrum von $(a_1, \cdots, a_n)$ bezeichneten Punktmenge des komplexen R^n analytisch sind. Diese nur skizzierte Theorie, die Resultate von H. Cartan verwendet, gilt für die Klasse der in A regulären Elemente a, d.h. solcher, für die $a-s$ für ge-

nügend großes komplexes s invertierbar ist und überdies die Funktion $(a-s)^{-1}$ in einer Umgebung von ∞ beschränkt bleibt. *G. Köthe* (Zbl 55, 107).

Waelbroeck, Lucien. Les algèbres à inverse continu. C. R. Acad. Sci. Paris 238 (1954), 640–641.

Verf. nennt eine lokalkonvexe, separierte, vollständige, kommutative komplexe Algebra A mit 1-Element „algèbre à inverse continu (à i.c.)", wenn eine Umgebung von 1 aus invertierbaren Elementen existiert und $a \to a^{-1}$ für $a=1$ stetig ist. I. Ist s ein maximales Ideal in einer Algebra A (à i.c.), so ist A/s der Körper der komplexen Zahlen. Das Spektrum $\mathrm{Sp}(a)$ für alle $a \in A$ ist eine nichtleere, kompakte Menge von komplexen Zahlen. Ist N ein vollständiges System von Halbnormen, so ist die „Spektralnorm"

$$h(a)=\sup_{n\in N}\limsup_{r\to\infty}[n(a^r)]^{1/r}=\max_{z\in \mathrm{Sp}(a)}|z|$$

eine stetige Halbnorm in A mit $h(ab)\leq h(a)\cdot h(b)$. Diese Resultate gestatten dem Verf. die Übertragung wichtiger Ergebnisse von Gelfand [Mat. Sb. N.S. 9(51) (1941), 3–24; MR 3, 51] über Banachalgebren auf Algebren à i.c. II. Die von E. R. Lorch [Trans. Amer. Math. Soc. 54 (1943), 414–425; MR 16, 834] für Funktionen in Banachalgebren definierten Begriffe der Ableitung und Holomorphie werden auf Funktionen in lokalkonvexen Algebren verallgemeinert und der vom Verf. entwickelte symbolische Kalkül [vorsteh. oder zweitfolg. Referat] wird zur Herleitung von Sätzen über im Sinne von Lorch holomorphe Funktion in einer Algebra à i.c. benutzt.

H. G. Tillmann (Zbl 56, 337).

Waelbroeck, Lucien. Structure des algèbres à inverse continu. C. R. Acad. Sci. Paris 238 (1954), 762–764.

A sei eine Algebra mit stetigen Inversen im Sinn von I. Gel'fand [Mat. Sb. N.S. 9(51) (1941), 3–24; MR 3, 51]. Es gilt der Satz, daß A algebraisch isomorph ist dem Quotientenring des Ringes der auf dem Spektrum von A analytischen Funktionen nach einem abgeschlossenen Ideal von Funktionen, die auf dem Spektrum verschwinden. Das Spektrum von A wird einmal im Sinn von Gelfand als der Raum der Maximalideale von A definiert, das andere Mal als projektiver Limes aus den Spektren je endlichvieler Elemente eines Systems von analytischen Erzeugenden von A im Sinn von Waelbroeck [siehe die oben referierte Arbeit]. *G. Köthe* (Zbl 55, 339).

Waelbroeck, L. Le calcul symbolique dans les algèbres commutatives. J. Math. Pures Appl. (9) 33 (1954), 147–186.

Sei A eine lokalkonvexe, separierte, kommutative Algebra mit Einselement über dem Körper der komplexen Zahlen. Für $a \in A$ definiert Verf. das Spektrum $\mathrm{sp}(a)$ als Teilmenge der Riemannschen Kugel: $s_0 \notin \mathrm{sp}(a)$, wenn es eine Umgebung $V(s_0)$ gibt, so daß $(a-s)^{-1}$ existiert für jedes $s \in V$ $(s\neq s_0)$ und die Menge dieser $(a-s)^{-1}$ beschränkt ist. $\mathrm{sp}(a)$ ist das Komplement des Holomorphiegebietes von $(a-s)^{-1}$ (Theorem 3). Verf. nennt a regulär wenn $\infty \notin \mathrm{sp}(a)$ und definiert das Spektrum eines Systems endlich vieler regulärer Elemente: $(s_1, \cdots, s_n) \in \mathrm{sp}(a_1, \cdots, a_n)$ wenn für jedes Polynom $p(z_1, \cdots, z_n)$, für das $\{p(a_1, \cdots, a_n)\}^{-1}$ existiert und regulär ist, $p(s_1, \cdots, s_n) = 0$ gilt (für $n=1$ äquivalent zur 1. Def.). $\mathrm{sp}(a_1, \cdots, a_n)$ ist kompakt und rationalkonvex im Sinne von H. Cartan und P. Thullen [Math. Ann. 106 (1932), 617–647]. Ist D_k beschränkte Umgebung von $\mathrm{sp}(a_k)$ $(k=1, \cdots, n)$ und $D=D_1\times\cdots\times D_n$, so vermittelt die Cauchysche Integralformel

$$f(z_1, \cdots, z_n)=(2\pi i)^{-n}\int_{\delta_1\times\cdots\times\delta_n}\cdots\int f(s_1 \cdots, s_n)\cdot (s_1-z_1)^{-1}\cdots(s_n-z_n)^{-1}ds_1\cdots ds_n$$

eine stetige Darstellung (Prop. 3)

$$f\to u(f)=(2\pi i)^{-n}\int_{\delta_1\times\cdots\times\delta_n}\cdots\int f(s_1, \cdots, s_n)\cdot (s_1-a_1)^{-1}\cdots(s_n-a_n)^{-1}ds_1\cdots ds_n\,[=f(a)]$$

der Algebra $H(D)$ der in D holomorphen Funktionen in A, wobei z_k auf a_k und die Konstante 1 auf das Einselement von A abgebildet wird [$H(D)$ mit Topologie der gleichmäßigen Konvergenz im Innern von D]. Unter Benutzung von Ergebnissen von H. Cartan [Ann. Sci. Ecole Norm. Sup. (3) 61 (1944), 149–197; Bull. Soc. Math. France 78 (1950), 29–64; MR 7, 290; 12, 172] und K. Oka [ibid. 78 (1950), 1–27; MR 12, 18] über Ideale analytischer Funktionen mehrerer Variablen zeigt Verf., daß diese Darstellung fortgesetzt werden kann zu einer stetigen Darstellung der Algebra $H(\mathrm{sp}(a_1, \cdots, a_n))$ aller auf $\mathrm{sp}(a_1, \cdots, a_n)$ analytischen Funktionen. $H(S)$ mit Topologie von G. Köthe [J. Reine Angew. Math. 191 (1953), 30–49; MR 15, 132], L. Van Hove [Bull. Soc. Math. Belg. 1952, 8–17; MR 15, 327]. $f(a_1, \cdots, a_n)$ wird definiert für alle auf $\mathrm{sp}(a_1, \cdots, a_n)$ analytischen Funktionen und kann explizit mit Hilfe der A. Weilschen Integralformel [Math. Ann. 111 (1935), 178–182] erhalten werden.

H. G. Tillmann (Zbl 56, 336).

Bade, William G. On Boolean algebras of projections and algebras of operators. Trans. Amer. Math. Soc. 80 (1955), 345–360.

This paper complements and extends a previous paper by the same author [Pacific J. Math. 4 (1954), 393–413; MR 16, 144]. Throughout this review B denotes a Boolean algebra of projections in a Banach space $\mathfrak{X}$. $\mathfrak{B}$ is called bounded if $|E|\leq C$ for some C and all E in $\mathfrak{B}$. It is called complete if for every family $\{E_\alpha\}$ in $\mathfrak{B}$: (a) $\mathfrak{X}$ is the direct sum of $\mathfrak{M}$ and $\mathfrak{N}$, where $\mathfrak{M}$ is the smallest closed linear manifold containing all the manifolds $E_\alpha\mathfrak{X}$ and $\mathfrak{N}$ is the intersection of all the manifolds $(I-E)_\alpha\mathfrak{X}$; (b) the projection E on $\mathfrak{M}$ along $\mathfrak{N}$ determined by the direct sum $\mathfrak{X}=\mathfrak{M}\oplus\mathfrak{N}$ is in $\mathfrak{B}$. If these conditions are fulfilled merely for all countable families, $\mathfrak{B}$ is called σ-complete. If $\mathfrak{B}$ is σ-complete, it is bounded. If $x \in \mathfrak{X}$, let $\mathfrak{M}(x)$ be the closed linear manifold spanned by Ex, E varying over $\mathfrak{B}$. Let $\overline{\mathfrak{B}}^s$ be the strong closure of $\mathfrak{B}$ in the algebra of all bounded linear operators on $\mathfrak{X}$. If $\mathfrak{B}$ is σ-complete, $\mathfrak{M}(x)$ is invariant under each member of $\mathfrak{B}$. Let $\mathfrak{B}(x)$ be the restrictions to $\mathfrak{M}(x)$ of the elements of $\mathfrak{B}$. Then $\mathfrak{B}(x)$ is a complete Boolean algebra of projections on $\mathfrak{M}(x)$. Proof of this depends on a result of Bartle, Dunford and Schwartz about countably additive vector-valued measures [Canad. J. Math. 7 (1955), 289–305; MR 16, 1123]. It is then proved that $\overline{\mathfrak{B}}^s$ is complete if $\mathfrak{B}$ is σ-complete. As a consequence, $\overline{\mathfrak{B}}^s$ can be shown to be complete if $\mathfrak{B}$ is bounded and $\mathfrak{X}$ is a weakly complete space. If $\mathfrak{B}$ is σ-complete and $x_0 \in \mathfrak{X}$, there exists $x_0^* \in \mathfrak{X}^*$ such that $x_0^*Ex_0\geq 0$ for each E in $\mathfrak{B}$, and $Ex_0=0$ whenever $x_0^*Ex_0=0$. A consequence is that the weak and strong operator topologies coincide on $\mathfrak{B}$ if $\mathfrak{B}$ is complete. An example shows that this need not be the case if $\mathfrak{B}$ is not σ-complete.

$\mathfrak{B}$ is said to have simple spectrum if $\mathfrak{X}=\mathfrak{M}(x_0)$ for some

x_0. Let $\mathfrak{A}(\mathfrak{B})$ be the algebra generated by $\mathfrak{B}$ in the uniform operator topology. If $\mathfrak{B}$ is complete and has simple spectrum, it is proved that $\mathfrak{A}(\mathfrak{B})$ is composed of all bounded linear operators which commute with $\mathfrak{B}$ (a result due to F. Wolf for the case of Hilbert space). Also, $\mathfrak{A}(\mathfrak{B})$ is composed of all bounded linear operators which leave invariant every linear manifold which is invariant under $\mathfrak{B}$. If $\mathfrak{B}$ is σ-complete, it is complete if and only if it is strongly closed; or, alternatively, if and only if $\mathfrak{A}(\mathfrak{B})$ is weakly closed. If $\mathfrak{X}$ is weakly complete it is enough here to assume that $\mathfrak{B}$ is bounded.

The paper concludes with applications to the theory of spectral operators of scalar type [in the terminology of Dunford, Pacific J. Math. **4** (1954), 321–354; MR **16**, 142]. If A is spectral, of scalar type and class $\mathfrak{X}^*$, with real spectrum, the weakly closed algebra generated by A consists of all extended bounded Baire functions of A.

A. E. Taylor (Los Angeles, Calif.).

Ogasawara, Tôzirô. A theorem on operator algebras. J. Sci. Hiroshima Univ. Ser. A. **18** (1955), 307–309.

The following theorem is proved: Let M be a B^*-algebra. For A, B positive in M: (a) if $A \geqq B$, then $A^{\frac{1}{2}} \geqq B^{\frac{1}{2}}$; (b) if $A \geqq B$ implies always $A^2 \geqq B^2$; then M is commutative. The author notes that (a) is a special case of a theorem of E. Heinz [Math. Ann. **123** (1951), 415–438; MR **13**, 471]. The proofs are short and (a) is generalized to include unbounded operators. [T. Kato has extended (a) by replacing $\frac{1}{2}$ by any number between 0 and 1, Math. Ann. **125** (1952), 208–212, Theorem 2, p. 209; MR **14**, 766.]

E. L. Griffin, Jr. (Ann Arbor, Mich.).

See also: Simboan, p. 468; Morse and Transue, p. 469; Calderón and Devinatz, p. 480; Zaslavskiĭ, p. 517.

Hilbert Space

Krein, M. G. On an application of the fixed-point principle in the theory of linear transformations of spaces with an indefinite metric. Amer. Math. Soc. Transl. (2) **1** (1955), 27–35.

Translated from Uspehi Mat. Nauk (N.S.) **5** (1950), no. 2(36), 180–190; MR **14**, 56.

Pinsker, M. S. The theory of curves in Hilbert space with stationary nth increments. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 319–344. (Russian)

The author and Yaglom developed independently the spectral and related properties of the curves described in the title, that is, in probability language, of processes with stationary nth increments. They outlined their results in two papers, one by Yaglom and Pinsker [Dokl. Akad. Nauk SSSR (N.S.) **90** (1953), 731–734; MR **15**, 238] and one by Pinsker and Yaglom [ibid. **94** (1954), 385–388; MR **15**, 806]. The present paper treats the topic by Hilbert-space methods, in non-probability language. Yaglom [Mat. Sb. N.S. **37(79)** (1955), 141–196; MR **17**, 167] treated the topic in probability language and corresponding methodology. The essential results are the extension of the spectral and prediction theories to the general case from the case $n=0$.

J. L. Doob.

★**Kato, Tosio. Quadratic forms in Hilbert spaces and asymptotic perturbation series.** Department of Mathematics, University of California, Berkeley, Calif., 1955. ii+145 pp. (mimeographed)

Let $H_\delta = H + \delta H^{(1)} + \delta^2 H^{(2)} + \cdots$ (the sum is formal) be a perturbation of self-adjoint operator H. Theorems are given in this report concerning the asymptotic validity of expansions like $\lambda_\delta = \lambda + \delta\lambda^{(1)} + \cdots$ for eigenvalues of H as perturbations of those of H. Operator H_δ is assumed semibounded and the Friedrichs extension method [Math. Ann. **109** (1934), 465–487] is systematically applied to extend symmetric operators. This method utilizes quadratic forms and a detailed expository account of such forms is included in the report. Previous work on asymptotic perturbation is covered and many extensions of earlier results are given. Relatively little knowledge of operator theory is assumed.

D. C. Kleinecke (Albuquerque, N.M.).

Gel'fand, I. M., and Fomin, S. V. Geodesic flows on manifolds of constant negative curvature. Amer. Math. Soc. Transl. (2) **1** (1955), 49–65.

Translated from Uspehi Mat. Nauk (N.S.) **7** (1952), no. 1(47), 118–137; MR **14**, 660.

Loomis, L. H. The lattice theoretic background of the dimension theory of operator algebras. Mem. Amer. Math. Soc. no. **18**, 36 pp. (1955).

In their classic papers on rings of operators in Hilbert space [cf., e.g., Ann. of Math. (2) **37** (1936), 116–229], Murray and von Neumann, among other things, deduced a dimension theory for certain lattices of closed linear subspaces. The present memoir gives a presentation of this dimension theory in abstract form, that is, starting from an abstract lattice L satisfying certain axioms. It is then shown in a very elegant manner that these axioms hold, not only for the Murray-von Neumann case, but more generally for any abstract C^*-algebra A possessing an existence property which the author formulates as follows:

(W): Let $\{a_\alpha\}$ be any family of elements in A with $\sup \|a_\alpha\|$ finite and $a_\alpha^* a_\beta = 0 = a_\alpha a_\beta^*$ for all $\alpha \neq \beta$; then a exists such that $ab = a_\beta b$ whenever $a_\alpha b = 0$ for all $\alpha \neq \beta$. (L then consists of all e satisfying $e^2 = e = e^*$ with $e \leqq f$ signifying $ef = e$.)

The author's axioms on L are: (1) L shall be complete, orthocomplemented and orthogonal-modular, i.e. $b(a \cup c) = a \cup bc$ whenever $a \leqq b$ and $a \perp c$. (2) L shall have an equivalence relation ($\sim$) satisfying: (A) $a \sim 0$ implies $a = 0$. (B) If $\{a_\alpha\}$ is an orthogonal family and $b \sim Ua_\alpha$, then $b = Ub_\alpha$ for some orthogonal family $\{b_\alpha\}$ with $b_\alpha \sim a_\alpha$ for every α. (C) (Complete additivity of $\sim$). If $\{a_\alpha\}$, $\{b_\alpha\}$ are orthogonal families with $a_\alpha \sim b_\alpha$ for every α, then $Ua_\alpha \sim Ub_\alpha$. (D) If a and b have a common complement then $a \sim b$ (in the C^*-algebra, $e \sim f$ signifies $a^*a = e$ and $aa^* = f$ for some a, and $e \perp f$ signifies $ef = 0$).

The dimension function is, of course, vector-valued in general, but real-valued if the lattice is irreducible relative to $\sim$ (a factor, in operator terminology). Many of the arguments used to construct the dimension function are, as the author says, classical. They go back in spirit to von Neumann's construction of Haar's measure on groups; and a detailed exposition for such a construction in reducible continuous geometry was given by von Neumann [in part reproduced in his lecture notes: Continuous geometry, Inst. Advanced Study, Princeton, 1937, Part III] and by T. Iwamura [Jap. J. Math. **19** (1944), 57–71; MR **8**, 35]. However the presentation given here permits infinite elements: it does not assume that every a is finite, i.e. $a \sim b$ and $b \leqq a$ imply $a = b$ (as is the case with continuous geometry). And much of the presentation uses only weak forms of (B) and (D).

The author's choice of axioms is admirably suited to his objective and for his C^*-algebras with property (W) the theory is elegant.

Moreover, with a few deft strokes, the author sketches extensions to include both a multiplicity theory for unitary representations of a locally compact group [due to G. Mackey, Ann. of Math. (2) **58** (1953), 193–221; MR **15,** 101] and also some work on classification of Boolean algebras under an equivalence relation [due to D. Maharam, Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 108–111; Trans. Amer. Math. Soc. **65** (1949), 279–330; MR **4,** 12; **10,** 519].

It is, however, unsatisfying that the presentation shuts out the case of reducible continuous geometries which do not possess an orthocomplementation. There (D) holds as the definition of $\sim$; (A) and (B) are obviously valid; the fact that $\sim$ is a congruence relation and that (B) holds can be shown directly (without dimension theory), if orthogonality is replaced by independence. This suggests the possibility of a more inclusive abstract dimension theory using a selected family of complements for each each element.

Also, the complete additivity of $\sim$ would be more pleasing as a theorem than as an axiom. Would it not be sufficient to postulate axiom (C) only for the special case that $\{a_\alpha \cup b_\alpha\}$ is an orthogonal family? *I. Halperin.*

Pukánszky, L. On the theory of quasi-unitary algebras. Acta Sci. Math. Szeged **16** (1955), 103–121.

A quasi-unitary algebra R [J. Dixmier, Comment. Math. Helv. **26** (1952), 275–322; MR **14,** 660] is an algebra over the complex numbers, on which an involutive antiautomorphism $x \to x^*$, an automorphism $x \to x^j$, and an inner product (x, y) are defined, such that R becomes a pre-Hilbert space satisfying the axioms (1) $(x^*, x^*)=(x, x)$, (2) $(x, x^j) \geqq 0$, (3) $(xy, z)=(y, x^{j*}z)$, (4) $x \to yx$ for fixed y is continuous, and (5) the linear set generated by elements of the form $xy+(xy)^j$ is dense in R for x, y, z arbitrary in R. The algebra is unitary if x^j always equals x.

Let U_x, V_x, S, J be the maximal closed extensions of the mappings $y \to xy$, $y \to yx$, s, j to the completion H of R. The weak closure of the operators U_x (resp. V_x) is a ring of operators R^g (resp. R^d). An element $a \in H$ is called left bounded, if there exists an operator V_a such that $V_a x = V_x a$ for every $x \in R$. If $J=[M'M^{-1}]$, where M commutes with R^g, $M \geqq 0$, and $M'=SMS$, then R^g is semi-finite (has no type III direct summand) and one obtains a canonical trace $\phi(A)=\sum_1^n (Ma_i; Mb_i)$ for elements A of R of the form $A=\sum V_{a_i} V_{b_i}^*$ (with a_i, b_i left bounded and in the domain of M).

The author extends the work of Dixmier, obtaining the following typical results: If R^g is semi-finite, then J has the form $[M' M^{-1}]$, where M satisfies the conditions above. If M_0' is a positive definite operator commuting with a semi-finite ring of operators N, and if ϕ is a trace defined on a two-sided ideal of N, then there exists a quasi-unitary algebra R with R^g *-isomorphic to N in such a way as to have M_0', ϕ correspond to the M' and cannonical trace of R^g. If R_1, R_2 are "maximal" (cannot be extended in such a way as to preserve j, s, etc.) with semi-finite R_1^g, R_2^g, and if there is a *-isomorphism between R_1^g, R_2^g with corresponding operators M_1', M_2' and maximal extensions of cannonical traces ϕ_1, ϕ_2, then R_1 and R_2 are isomorphic. (These results extend work of Godement [Ann. of Math. (2) **59** (1954), 47–62; MR **15,** 441] and Segal [ibid. **57** (1953), 401–457; MR **14,** 991]. The author concludes by giving an example of a quasi-unitary algebra with semi-finite R^g for which the cannonical trace is not maximal for any choice of M, and a condition for R^g to be finite. *E. L. Griffin, Jr.* (Ann Arbor, Mich.).

See also: Indritz, p. 478; Goldman, p. 512; Rankin, p. 523.

TOPOLOGY

Inagaki, Takeshi. Une note sur l'axiome (T_5) de séparation. Math. J. Okayama Univ. **5** (1955), 1–3.

The author obtains two conditions which are equivalent to axiom T_5 (=completely normal) for topological spaces, and uses one of these to obtain a necessary and sufficient condition for a subset of a T_5-space to have Urysohn-Menger dimension $\leqq n+1$ at a point. *E. Michael.*

Iséki, Kiyoshi. On a conjecture of K. Nagami. Proc. Japan Acad. **31** (1955), 430.

The author gives a negative answer to a conjecture of K. Nagami [Nagoya Math. J. **8** (1955), 69–70; MR **16,** 946]: Is every completely paracompact Hausdorff space necessarily perfectly normal?

Papić, Pavle. Sur une classe d'espaces abstraits. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9** (1954), 197–216. (Serbo-Croatian summary)

Appelons espace général-topologique tout système $(P, \mathcal{S})$ constitué par un ensemble quelconque P d'éléments appelés points et par une loi quelconque $\mathcal{S}$ suivant laquelle un point quelconque $a \in P$ est ou non contigu à un ensemble quelconque $E \subset P$.

Un espace général-topologique $(P, \mathcal{S})$ est dit [d'après A. Appert et K. Fan, Espaces topologiques intermédiaires···, Hermann, Paris, 1951, p. 14; MR **13,** 54] admettre une base B si B est une famille d'ensembles, dont la réunion est P, et si pour tout point $a \in P$ et pour tout ensemble $E \subset P$, on a: (a contigu à E) $\Leftrightarrow$ (chaque ensemble de B qui comprend a, comprend au moins un point de E).

L'A, appelle (p. 210) espace Π tout espace général-topologique admettant une base B telle que pour tous ensembles $O_1 \in B$ et $O_2 \in B$ on ait toujours soit $O_1 \subset O_2$, soit $O_2 \subset O_1$, soit O_1 disjoint de O_2. Il appelle espace R [notion due à G. Kurepa, Publ. Math. Univ. Belgrade **5** (1936), 124–132] tout espace Π vérifiant l'axiome T_1 (de M. Fréchet), lequel revient ici à dire que pour tous points distincts $a \in P$ et $b \in P$, il existe un ensemble de B qui comprend a sans comprendre b. Tout espace R est un espace accessible au sens de M. Fréchet. Une base B sera toujours considérée comme partiellement ordonné par la relation $\supset$ inverse de l'inclusion.

On démontre par induction transfinie que tout espace R admet une base ramifiée de voisinages, entendant par là une base B qui est un tableau ramifié au sens de G. Kurepa [Thèse, Paris, 1935, p. 73] c'est-à-dire qui est telle que l'ensemble de tous les éléments de B qui contiennent un élément arbitrairement donné de B est bien ordonné. Lorsque B est une base ramifiée de voisinages d'un espace R, on appelle 1ère rangée $R_1 B$ de B l'ensemble de tous les éléments de B sans prédécesseur dans B; de plus, pour tout ordinal $\alpha > 1$, on appelle α-ième rangée $R_\alpha B$ de B l'ensemble $R_1(B - \bigcup_{1 \leq t < \alpha} R_t B)$; et on appelle

rang γB de B le plus petit ordinal $\gamma \geqq 1$ tel que la γ-ième rangée de B soit vide.

L'A. établit de nombreuses propriétés des espaces R parmi lesquelles nous citerons les suivantes (pour la terminologie voir A. Appert et K. Fan, loc. cit.): Pour qu'un espace R soit distanciable, il faut et il suffit que cet espace admette une base ramifiée de voisinages dont le rang soit $\leqq \omega_0$. Pour qu'un espace distanciable soit un espace R, il faut et il suffit qu'on puisse y définir une distance $\varrho(a, b)$ (compatible avec la topologie de l'espace) et vérifiant la condition suivante: pour tous points a, b, c de l'espace et pour tout nombre réel $\varepsilon > 0$, on a: $\varrho(a, b) < \varepsilon$ et $\varrho(b, c) < \varepsilon$ entraînent $\varrho(a, c) < \varepsilon$. Tout espace R localement distanciable est distanciable (le terme „localement distanciable" signifie „dont chaque point possède un entourage qui est un sous-espace distanciable"). Tout espace R localement séparable est distanciable. Tout espace R localement compact est distanciable. Pour qu'un espace R soit compact, il faut et il suffit que chaque base ramifiée de voisinages de cet espace ait un rang $\leqq \omega_0$. Tout espace R est paracompact [au sens de J. Dieudonné, J. Math. Pures Appl. (9) **23** (1944), 65–76; MR **7**, 134]. Tout espace R distanciable est homéomorphe à un espace totalement ordonné [au sens de G. Kurepa, Thèse, p. 11].

L'A. étudie les espaces $T(F)$ introduits par G. Kurepa en même temps que les espaces R (loc. cit.) et qui peuvent se définir comme étant les espaces R à caractère dénombrable et admettant une base ramifiée F de voisinages telle que chaque sous-famille disjonctive (c'est-à-dire formée d'ensembles deux à disjoints) d'éléments de F soit dénombrable. Il établit diverses conditions de distanciabilité des espaces $T(F)$. Les espaces $T(F)$ sont intimement liés au problème connu de Suslin. Tout espace R vérifie les six axiomes topologiques de séparation T_i ($i=0, 1, \cdots, 5$); autrement dit, dans un espace Π l'axiome T_1 entraîne tous les autres. Tout espace R est donc complètement normal. Tout espace Π est un espace $(\mathcal{U}_\alpha D)$ vérifiant T_5 et donc aussi T_4. L'A. donne un exemple d'espace Π vérifiant T_3 sans vérifier l'axiome T_0 de Kolmogoroff, et un exemple d'espace Π vérifiant T_0 sans vérifier T_3. Dans tout espace Π, on a: (T_0 et T_3) $\Leftrightarrow T_1$.

L'A. étudie enfin sous le nom d'espaces R_f les espaces R admettant une base ramifiée B de voisinages telle que la réunion de tous les éléments d'une rangée quelconque de B est un ensemble fermé. Le rapporteur pose la question suivante: comment et sous quelles formes les propriétés des espaces R s'étendent-elles aux espaces Π.

A. Appert (Angers).

Papić, Pavle. Sur les espaces pseudo-distanciés. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9** (1954), 217–228. (Serbo-Croatian summary)

L'A. étudie les „espaces pseudo-distanciés" de G. Kurepa [C.R. Acad. Sci. Paris **198** (1934), 1563–1565] appelés par M. Fréchet [Portugal. Math. **5** (1946), 121–131; MR **8**, 48] „espaces à écart abstrait symétrique et régulier", et par nous-même [A. Appert et K. Fan, Espaces topologiques intermédiaires···, Hermann, Paris, 1951, p. 140; MR **13**, 54] „espaces de Kurepa-Fréchet". Tout espace pseudo-distancié qui n'est pas distanciable est un espace R. Etude des notions de convergence, d'ensemble pseudo-compact, de suite de Cauchy, et d'espace complet introduites par M. Fréchet (loc. cit.) dans les espaces pseudo-distanciés. Etude des espaces $E(M)$ de G. Kurepa [C.R. Acad. Sci. Paris **203** (1936), 1049–1052]. Réponse négative à une question posée par G. Kurepa dans la note précédente.

A. Appert.

Pitcairn, Joel. The completion of a uniform space. J. London Math. Soc. **31** (1956), 119–121.

It is shown that the completion of a Hausdorff uniform space X can be obtained as follows: Let X^* be the space of non-void subsets of X, provided with the natural uniformity, and define $(X^*)^*$ similarly. Define $\pi: X \to (X^*)^*$ by $\pi(x) = \{A \in X^* | x \in A\}$; then π is a uniform homeomorphism into, and the closure of π is complete.

E. Michael (Seattle, Wash.).

Tamari, D. On a generalization of uniform structures and spaces. Bull. Res. Council Israel **3** (1954), 417–428.

Remarques élémentaires sur la structure qu'on obtient en omettant l'axiome de symétrie dans la définition des structures uniformes (c'est-à-dire l'axiome affirmant que si U est un entourage, il en est de même de U^{-1}).

J. Dieudonné (Zbl **56**, 415).

Grünbaum, Branko. A characterization of compact metric spaces. Riveon Lematematika **9** (1955), 70–71. (Hebrew. English summary)

A metric space is hypernormal if the distance between any pair of disjoint closed sets is positive. [The term hypernormal has been used in a different sense by other authors.] It is proved that every hypernormal space is complete, and that if it is not compact then there exist $r > 0$ and an infinite set of points each of which is at a distance greater than r from its complement in the whole space.

M. Jerison (Lafayette, Ind.).

Knaster, B., Mioduszewski, J., et Urbanik, K. Points-limites et points de continuité. Colloq. Math. **3** (1955), 164–169.

Soit d'abord une fonction $f = f(\xi)$ définie sur un ensemble E de points d'un espace distancié $\mathfrak{X}$ et prenant ses valeurs dans un espace distancié $\mathfrak{Y}$. Un point $x \in \mathfrak{X}$ est dit point de continuité de f si $x \in E$ et si f est continue en x. Un point $x \in \mathfrak{X}$ est dit point-limite de f si $x \in \bar{E}$=fermeture de E, et si, pour tout nombre $\varepsilon > 0$, il existe un ensemble ouvert $U \subset \mathfrak{X}$ tel que $x \in U$ et que le diamètre de

$$f(E \cap U - \{x\})$$

soit $< \varepsilon$. Un point $x \in \mathfrak{X}$ est dit point-limite unilatéral de f s'il existe un ensemble ouvert $V \subset \mathfrak{X}$ tel que x soit un point-limite de la fonction partielle $f|E \cap V$ (on désigne ainsi la restriction de f à l'ensemble $E \cap V$).

L'A. compare ces trois notions dont chacune est plus restrictive que la suivante; il démontre, entre autres, le théorème suivant: Pour qu'une fonction f définie sur la totalité d'un espace distancié complet $\mathfrak{X}$ et prenant ses valeurs dans un espace distancié séparable $\mathfrak{Y}$ soit de 1ère classe de Baire, il faut et il suffit que, pour tout ensemble parfait $F \subset \mathfrak{X}$, la fonction partielle $f|F$ ait un point-limite unilatéral $x \in F$. L'A. pose la question de savoir si la condition ci-dessus reste suffisante quand $\mathfrak{X}$ n'est pas complet, les autres hypothèses restant les mêmes.

A. Appert (Angers).

Klee, V. L., Jr. Some finite-dimensional affine topological spaces. Portugal. Math. **14** (1955), 27–30.

Two topologies are described in m-dimensional real Euclidean space E^m ($m \geqq 2$), in each of which E^m is an affine topological space in the sense of Fréchet [Les espaces abstraits, Gauthier-Villars, Paris, 1928] but not a linear

topological space. One of these topologies was previously introduced by the author [Duke Math. J. 18 (1951), 443–466; MR 13, 354]. Various elementary topological properties of the spaces are discussed; typical results proved are that in one topology E^m is not regular, while in the other it is completely regular but not normal.

J. H. Williamson (Princeton, N.J.).

Calabi, Lorenzo. Due nuove strutture topologiche per insiemi di funzioni locali. Rend. Mat. e Appl. (5) **14** (1955), 581–593.

When the set of local analytic functions is topologized in the manner of Fantappié, it fails to satisfy the (T_1) separation axiom. The main purpose of the present paper is to enlarge the topology in order to gain more separation properties. For a compact infinite metric space X, let C denote the set of all continuous complex-valued functions u having domain D_u a nonempty open subset of X. The set C is topologized in three different ways, giving rise to spaces C_F, C_s, and C_{ss}, by taking as a fundamental system of neighborhoods of u the class of all sets of the form $(A, \sigma)_u$, $\{A, \sigma, \tau\}_u$, and $(A, \sigma, \tau)_u$, respectively, where A is a compact subset of D_u, $\sigma>0<\tau$,

$$(A, \sigma)_u=\{v: A\subset D_v \& \|u-v\|_A<\sigma\},$$

$$\{A, \sigma, \tau\}_u=\{v: A\subset D_v \& \|u-v\|_A<\sigma \& s(D_u, D_v)<\tau\},$$

$$(A, \sigma, \tau)_u=\{v: A\subset D_v \& \|u-v\|_A<\sigma \& s(D_u', D_v')<\tau\},$$

and s is the Hausdorff distance. The space C_F is analogous to that of Fantappié, and is shown to admit a "strict local order"; spaces which admit such orders are investigated. The space C_s still violates the (T_1) separation axiom, but C_{ss} turns out to be separable and metrizable. A set $L\subset C$ is called "linear" provided $tu \in L$ and

$$u+v|D_u\cup D_v \in L$$

whenever t is a complex number and u, $v \in L$. The open linear subsets of C_s and C_{ss} are characterized. [Reviewer's notes: 1) In the author's discussion, X is the complex sphere; however, the results appear to carry through as stated above. 2) The author observes that, in a certain uniformity, C_{ss} fails to be complete. It may be shown further that C_{ss} is of the first category in itself, as is its subspace consisting of all local analytic functions.]

V. L. Klee (Los Angeles, Calif.).

Ribeiro de Albuquerque, José. Propriétés de connexion dans les espaces abstraits. Univ. Lisboa. Revista Fac. Ci. A. (2) **3** (1954–1955), 333–350.

This is a summary of the author's thesis which will be published later in the same journal. No proof of the announced results is given here. In a topological space S, let Γ_1 denote the family of all pairs (X, Y) of mutually connected subsets of S, i.e. the family of all pairs (X, Y) such that $(X'\cap Y)\cup(X\cap Y')\neq 0$ [where X' denotes the derived set of X]. Let Γ_2 denote the family of all pairs (X, Y) of mutually chained subsets of S, i.e. the family of all pairs (X, Y) such that $(X'\cap Y)\cup(X\cap Y')\cup(X'\cap Y')\neq 0$. Let Δ_1 denote the family of all connected sets in S, and Δ_2 denote the family of all well-chained sets (i.e., sets with connected closures). By taking one of the concepts "mutually connected", "mutually chained", "connected", "well-chained" as an undefined concept of topology, the author studies sets of axioms for the families Γ_1, Γ_2, Δ_1, Δ_2. Previously, A. D. Wallace [Ann. of Math. (2) **42** (1941), 687–697; An. Acad. Brasil. Ci. **14** (1942), 203–206; MR 3, 37; 4, 87] and P. Szymanski [Mathematica, Timişoara **17** (1941), 65–84; MR 4, 87] have given sets of axioms for topology based upon the concept of separation as an undefined property.

Ky Fan (Notre Dame, Ind.).

Zaslavskiĭ, I. D. Some criteria of compactness in metric and normed spaces. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 953–956. (Russian)

Basic theorem: Suppose that α and the α_t's are pseudometrics on the set M, that α_t converges pointwise to α, and that always $\alpha_t\leqq\alpha$. Then M is totally bounded under α if and only if the convergence is uniform and M is totally bounded under each α_t. This is applied to yield compactness criteria for subsets of various normed linear spaces (C, L^p, M, and Lip on $[a, b]$). For example, a subset M of $C[a, b]$ is totally bounded if and only if the following two statements are true: (i) M is bounded; (ii) for each $\varepsilon>0$ there are numbers t_i with $a=t_0<t_1<\cdots<t_n=b$ such that $\max_{t\in(a,b)}|x(t)|-\max_{1\leqq i\leqq n}|x(t_i)|<\varepsilon$ for all $x \in M-M$.

V. L. Klee (Los Angeles, Calif.).

Hsu, L. C. Concerning derived limits and set sequences in complete metric spaces. J. Chinese Math. Soc. (N.S.) **1** (1951), 88–97. (Chinese summary)

The author states four theorems concerning sequences of points or sequences of bounded point sets in a complete metric space X. His first theorem and the principal lemma for it are both false as easy exemples using sequences of points with repetitions demonstrate. His second theorem, topologizing the class of non-null bounded open sets by means of the Hausdorff metric, is well known.

In his definitions and a remark the author seems to assume that for any sequence $\{X_i\}$ of bounded sets in X there must be a limit point of some sequence $\{x_{\nu_n}\}$, where $x_{\nu_n}\in X_{\nu_n}$ and for each n, $\nu_n<\nu_{n+1}$. His third and fourth theorems are inadequately stated, due in part to this apparent assumption, but are essentially valid. Theorem 3 states: If the sequence $X_{n1}, X_{n2}, X_{n3}, \cdots$ of bounded point sets in X converges to $X_{n\infty}$ uniformly with respect to n, then we have

$$\lim_{n\to\infty}(\lim_{m\to\infty} X_{nm})=\lim_{m\to\infty}(\lim_{n\to\infty} X_{nm}).$$

Definition 4.2 states: The sequence X_{n1}, X_{n2}, X_{n3} is said to converge to the limiting element $X_{n\infty}$ uniformly with respect to n, if for every given $\varepsilon>0$, there exists a large number N (depending on ε only) such that, for all n, $\|X_{nj}-X_{n\infty}\|<\varepsilon$ whenever $j\geqq N=N(\varepsilon)$. Theorem 4, stated without detailed proof, asserts that under the hypothesis of Theorem 3

$$\lim_{n\to\infty}(\lim_{m\to\infty} X_{nm})=\lim_{n\to\infty} X_{nn}=X_{\infty\infty}.$$

R. D. Anderson (Princeton, N.J.).

Gravett, K. A. H. A characterization of frontier. Proc. Cambridge Philos. Soc. **52** (1956), 152–153.

In this note conditions on a transformation f of P, the set of all subsets of a space X, into P are given which are equivalent to the existence of a topology for X such that for each $s \in P$, $f(s)$ is the frontier of s.

E. Dyer.

Smythe, William R., Jr. A theorem on upper semi-continuous decompositions. Duke Math. J. **22** (1955), 485–495.

The author proves the following theorem. Let M be a complete, locally compact, separable, metric space and G a topologically contracting collection which is an upper semi-continuous decomposition of M. Suppose that given

any non-degenerate member C of G, any open set U containing C, and any $\varepsilon>0$, there exists a point $p \in C$ and a mapping g of M onto itself such that (i) $g(C)=p$, (ii) g maps $M-C$ homeomorphically onto $M-p$, (iii) g is stationary outside of U, and (iv) $\delta[g(D)]<\delta(D)+\varepsilon$ for each $D \in G$, where $\delta(S)$=diameter of S. Then M and $D(M, G)$ are homeomorphic.

The author shows as a non-trivial consequence of this theorem that: If G is an upper semi-continuous decomposition of Euclidean n-space whose non-degenerate members are strictly convex, then the decomposition space is homeomorphic to Euclidean n-space. A compact set C is said to be strictly convex if every segment joining two of its points is contained, except possibly for its end points, in the interior of C. *R. D. Anderson.*

Hamstrom, Mary-Elizabeth. A characterization of plane continuous curves which are filled up by continuous collections of continuous curves. Amer. J. Math. 77 (1955), 914–928.

The author states and proves a theorem asserting a result as in the title. The characterization is rather complicated, employing six conditions on a plane continuous curve, two of such conditions involving technical prior definitions. The reviewer has considered the problem of the characterization at some length. It seems likely that no appreciably simpler characterization than that given is possible at this time. Furthermore, given a continuous curve in the plane, it is usually not difficult to determine whether or not the author's conditions are satisfied. *R. D. Anderson* (Princeton, N.J.).

Floyd, E. E. Real-valued mappings of spheres. Proc. Amer. Math. Soc. 6 (1955), 957–959.

The author proved that for any continuous mapping from 2-sphere to the real line, there exist three points x_0, x_1, x_2, congruent to given three points y_0, y_1, y_2 such that all the x_i's are mapped to a single point on the real line. This is proved by finding the rotation which maps y_i's to x_i's. This is a generalization of a theorem by Kakutani [Ann. of Math. (2) 43 (1942), 739–741; MR 4, 111]. *H. Yamabe* (Minneapolis, Minn.).

Hanai, Sitiro. On quasi-interior mappings. Math. Japon. 3 (1955), 117–120.

As defined by G. T. Whyburn [Mem. Amer. Math. Soc. no. 1 (1950); MR 13, 764] a quasi-interior mapping of a T_1-space S onto a T_1-space E is a mapping f such that if $p \in E$ and U is an open set in S containing a compact component of $f^{-1}(p)$, then p is interior to $f(U)$. The author shows that certain topological properties are invariant under such maps. For example: if f is quasi-interior and for each $p \in E$, $f^{-1}(p)$ is compact, then each of the conditions: local connectedness, local compactness, and perfect separability, is invariant under f; if f is quasi-interior, each $f^{-1}(p)$ is a compact continuum, S is locally connected and E is connected, then each component of S maps onto E under f. Regarding the last result it might be mentioned that even without the local connectedness on S, there is only one component of S. *E. Dyer.*

Whitney, Hassler. On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane. Ann. of Math. (2) 62 (1955), 374–410.

This paper is primarily devoted to the singularities of maps of the plane E_2 into itself.

In Part A the author selects a class of such maps, termed "excellent", whose singularities are especially transparent. He then shows in Part B that every r-smooth (of class C^r) map $f:E_2 \to E_2$, can be arbitrarily well approximated up to r derivatives by an excellent map. The notion of excellent map is therefore the precise generalization of a nondegenerate function of E_2. The singular points of a smooth map, $f:E \to E$ are of course the points at which the Jacobian J of f vanishes. If f is 2-smooth, and the differential of its Jacobian does not vanish at any singular point, i.e. $J(p)=0 \Rightarrow dJ \neq 0$ at p, then f is termed "good". The singular points of a good map are easily seen to be smooth curves in E_2. Finally, f is excellent if a) f is of class 3 and good, b) if $\phi(t)$ is a 2-smooth parametrisation of a singular curve near $p=\phi(0)$; then: either $df \circ \phi(t)/dt|_{t=0} \neq 0$ or if this expression vanishes, $d^2 f \circ \phi(t)/dt^2|_{t=0} \neq 0$. The singular point p is called a fold point in the first case, a cusp point in the second case. In Part C the author gives normal forms for a sufficiently smooth excellent map near fold and cusp points. Thus at a fold point, p, f is given (in terms of suitable coordinates (x, y) in E_2, (u, v) in E_2', $p \sim x=0$, $y=0$) by: $u=x^2$, $v=y$. At a cusp-point by: $u=xy-x^3$, $v=y$.

The generic character of an excellent map is proved in Part B by methods which the author initiated in his earlier papers [e.g., Ann. of Math. (2) 37 (1936), 645–680]. However, the account here is completely selfcontained. He first recasts the approximating problem in the following fashion: Suppose $g:R \to \bar{E}$ is a map of an open set, $R \subset E_k$, into $\bar{E}$; and suppose $Q \subset R$ and $T \subset \bar{E}$ are compact subsets. The problem is to find a map g' approximating g arbitrarily well for which $g'(Q) \cap T=0$. The author tackles this question by first measuring the "width" of the sets Q and T by means of his notion of finite μ-extent. (μ-extent is analogous to Hausdorff μ-measure.) Then in terms of this concept his answer is: If $Q[T]$ have finite K-extent [M-extent] and if g can be imbedded in a ν-parameter family of maps of R into $\bar{E}$, with $\nu>K+M$, then there always exists a g' with the desired properties.

The paper closes with some topological considerations. It is shown for instance, that while cusp points can be locally eliminated by arbitrarily good approximations, fold points may not. The author also finds algebraic relations between the singularities of an excellent map. As an example we cite: If f and g are excellent maps which agree in a neighbourhood of a simple closed curve A, which contains no cusp points, then f and g have the same number mod 2 of cusp points inside A. As an application it is shown that there is no excellent map of the projective plane into the plane without cusp points. *R. Bott.*

Bajada, E. Sulla teoria dei punti critici e la topologia delle varietà. Matematiche, Catania 8 (1953), no. 1, 27–49.

Darstellung der Morseschen Theorie der kritischen Punkte auf kompakten differenzierbaren Mannigfaltigkeiten, insbesondere Herleitung der Morseschen Beziehungen zwischen den Zusammenhangszahlen der Mannigfaltigkeit und den Typenzahlen kritscher Punkte.
H. Seifert (Zbl 52, 200).

Dowker, C. H. Homotopy extension theorems. Proc. London Math. Soc. (3) 6 (1956), 100–116.

Given X and a closed $A \subset X$, let $H=X \times 0 \cup A \times I$, I the unit interval. Let (X, A, Y, f) denote a (continuous) $f:H \to Y$, and call (X, A, Y, f) extensible if f has a (continuous) extension $F:X \times I \to Y$. The author gives very

general conditions under which (X, A, Y, f) is extensible. Main theorems: (1) (X, A, Y, f) is extensible whenever X is countably paracompact normal, Y metric, A a G_δ, and $f|H\cap\{X\times(0\leq t<1)\}$ has an extension over a neighborhood in $X\times(0\leq t<1)$. (2) (X, A, Y, f) is extensible for every f whenever X is countably paracompact collectionwise normal, Y a metric ANR, and either A is a G_δ or Y is an absolute G_δ. Of independent interest is the following lemma needed in the proof: If S is countably paracompact collectionwise normal, and T is compact metric, then $S\times T$ is collectionwise normal. (3) For normal X, (X, A, Y, f) is extensible for all Y and all f if and only if A is a G_δ and also a neighborhood deformation retract of X. Several applications of the results are included.

J. Dugundji (Los Angeles, Calif.).

Aleksandrov, P. S. The present status of the theory of dimension. Amer. Math. Soc. Transl. (2) **1** (1955), 1–26.

Translated from Uspehi Mat. Nauk (N.S.) **6** (1951), no. 5(45), 43–68; MR **13**, 764.

Aigner, Alexander. Einige Sätze über Lagebeziehungen in endlichen Graphen. J. Reine Angew. Math. **195** (1956), 18–21 (1955).

In this paper a graph is a not necessarily connected set of points and lines. Every line joins two distinct points and any pair of distinct points is joined by either 0 or 1 line. An even (odd) solution is a nonempty subset of the points which is joined to any point of the graph by an even (odd) number of lines ("no lines" counting as even). The even solutions form an additive group under mod 2 addition and it is shown that the number of generators is congruent mod 2 to the number of points of the graph. In particular there is always an even solution if the number of points is odd. Finally there is an odd solution when, and only when, there are no even solutions with an odd number of points. The investigation was motivated by a problem in number theory [Aigner, same vol., 3–17; MR **17**, 464]. *J. W. S. Cassels.*

See also: Floyd, p. 450; Haupt und Pauc, p. 469.

Algebraic Topology

Nakaoka, Minoru. Homotopy of two-fold symmetric products of spheres. J. Inst. Polytech. Osaka City Univ. Ser. A. **6** (1955), 19–30.

Utilisant des résultats de R. Bott [Ann. of Math. (2) **57** (1953), 579–590; MR **15**, 54], S. D. Liao [Trans. Amer. Math. Soc. **77** (1954), 520–551; MR **16**, 504] et S. K. Stein [Ann. of Math. **59** (1954), 570–583; MR **15**, 890] sur l'homologie et la cohomologie du produit symétrique S_n*S_n de deux n-sphères, l'auteur calcule des groupes d'homotopie $\pi_i(S_n*S_n)$. Ils sont stables pour $i\leq 2n-2$; les groupes stables, pour $n+1\leq i\leq n+9$, sont: $\pi_{n+3}=Z_3$, $\pi_{n+5}=Z_2$, $\pi_{n+7}=Z_{15}$, $\pi_{n+8}=Z_2$, $\pi_{n+9}=Z_2$, les autres sont nuls. De plus $\pi_i(S_n*S_n)$ est fini pour tout $i\neq n$, et a même composante p-primaire que $\pi_i(S_n)$ pour p premier impair. Ces résultats obtenus par deux voies: l'une utilise les fibrations de Cartan-Serre-Whitehead, les résultats de Serre sur la cohomologie d'Eilenberg-MacLane mod 2 et les relations d'Adem; l'autre utilise un complexe réduit ayant même type d'homotopie que S_n*S_n en dimensions $\leq n+6$.

Dans une section finale, l'auteur considère les produits cycliques $\vartheta_{n,p}$ de p sphères S_n; il annonce (sans démonstration) que pour p premier impair, la composante p-primaire de $\pi_i(\vartheta_{n,p})$ est Z_p pour $i=n+2j$ $(1\leq j\leq p-2)$ et $i=n+2p-3$, est nulle pour les autres valeurs de $i\leq n+2p-2$. Il donne aussi des résultats sur les carrés de Steenrod dans $H*(Y*Y; Z_2)$ et calcule les groupes stables $\pi_i(Y*Y)$ $(i\leq 2n-2)$ pour $i\leq n+9$, en notant Y la $(n-1)$-ième suspension du plan projectif réel. *H. Cartan.*

Dold, Albrecht. Über fasernweise Homotopieäquivalenz von Faserräumen. Math. Z. **62** (1955), 111–136.

This paper is concerned with the classification of fibre bundles [Steenrod, The topology of fibre bundles, Princeton, 1951; MR **12**, 522] according to fibre homotopy equivalence. Two fibre bundles B_1 and B_2 over the same base space are fibre homotopy equivalent [R. Thom, Ann. Sci. Ecole Norm. Sup. (3) **69** (1952), 109–182; MR **14**, 1004] if there are maps $\{h_1: B_1\to B_2\}$ and $\{h_2: B_2\to B_1\}$ lying over the identity, and such that the composite maps h_2h_1 and h_1h_2 can be deformed to the identity map of B_1, resp. B_2, by a deformation with the property that each point in its fibre during the deformation.

The author starts by showing that two fibre bundles over a polyhedron are fibre homotopy equivalent if and only if there is a fibre-preserving map of one into the other which lies over the identity, and which when confined to a single fibre is an ordinary homotopy equivalence. After studying the problem of fibre homotopy equivalence in general, he then starts to study the special case wherein the base space is assumed to be a sphere.

Suppose now that Y is a locally compact topological space, and G is a group of transformations of Y. The ordinary classification of fibre bundles over the m-dimensional sphere S^m with fibre Y, and structural group G says that there is a one-to-one correspondence between equivalence classes of such bundles and conjugacy classes of elements of $\pi_{m-1}(G)$ under the action of $\pi_0(G)$. If G^* denotes the space of homotopy equivalences of Y with itself, then it is shown that G^* plays the role of the structural group G in the classification of fibre bundles over S^m with fibre Y by fibre homotopy equivalence. In particular there is a natural map $j: G\to G^*$ which induces maps

$$j^\#: \pi_{m-1}(G)\to\pi_{m-1}(G^*) \text{ and } j^\#: \pi_0(G)\to\pi_0(G^*).$$

Further, the fibre bundles over S^m corresponding to the conjugacy classes of α, $\beta\in\pi_{m-1}(G)$ are fibre homotopy equivalent if and only if $j^\#(\alpha)$, $j^\#(\beta)\in\pi_{m-1}(G^*)$ belong to the same conjugacy class under the action of $\pi_0(G^*)$.

The preceding results are used to show that a single fibre homotopy equivalence class of fibre bundles over S^m may contain infinitely many non-equivalent fibre bundles. For example, if one considers bundles with fibre S^n, and structural group the orthogonal group O_{n+1}, then if $m=0 \pmod 4$, and $n>\frac{1}{2}m$ the kernel of $\pi_{m-1}(O_{n+1})\to\pi_{m-1}(G^*)$ has rank 1.

In addition to the results just outlined, this paper contains many interesting examples. *J. C. Moore.*

Hurewicz, Witold. On the concept of fiber space. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 956–961.

Let $p: E\to B$ be a continuous map and let

$$\Omega_p=\{(\varepsilon, \omega)\in E\times B^I \mid \omega(0)=\varepsilon\}.$$

Let $\tilde{p}: E^I\to\Omega_p$ be defined by $\tilde{p}\tau=(\tau(0), p\tau)$. Then the triple (E, B, p) is defined to be a fiber space if $\tilde{p}$ has a cross section. It is shown that (E, B, p) is a fiber space if

and only if the covering homotopy condition is satisfied for it. A fiber space (E, B, p) is called regular if $\bar{p}$ admits a cross section λ such that whenever ω is a constant path in B then $\lambda(e, \omega)$ is a constant path in E. A fiber space is regular if and only if it satisfies the covering homotopy condition with the covering homotopy constant when the base homotopy is constant. It is shown that every fiber space over a metric B is regular.

The triple (E, B, p) is called a local fiber space if for every $b \in B$ there exists a neighborhood U of b such that the triple $(p^{-1}(U), U, p|p^{-1}(U))$ is a fiber space. The main theorem asserts that a (regular) local fiber space over a paracompact space B is a (regular) fiber space in the large. It follows from this that a local fiber space satisfies the covering homotopy condition for maps of paracompact spaces.

Lastly the author compares his definition of fiber space with other definitions. A fiber space in the sense of Hu [Proc. Amer. Math. Soc. **1** (1950), 756–762; MR **12**, 435] is a local regular fiber space, so if the base B is a paracompact it is a regular fiber space. If the base B is a metric absolute neighborhood retract, the two concepts coincide and agree with the concept of fiber space in the sense of Hurewicz and Steenrod [Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 60–64; MR **2**, 323]. *E. H. Spanier.*

Hurewicz, Witold, and Fadell, Edward. On the spectral sequence of a fiber space. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 961–964.

Let (X, B, π) be a regular fiber space (see preceding review for definition) and let $F=\pi^{-1}(b)$ be the fiber over a fixed $b \in B$. It is shown that if $\pi_i(B)=0$ for $i \leqq r$, then in the spectral sequence for singular homology associated to (X, B, π) the first $r+1$ terms $E_1, E_2, \cdots, E_{r+1}$ and the first r of the corresponding differentials $d_1, d_2, \cdots, d_r$ are completely determined by B and F. Comparing with the case $X=B\times F$ it follows that $E_i=E_2$ for $2 \leqq i \leqq r+1$ and $d_i=0$ for $2 \leqq i \leqq r$. The proof depends on the fact that because of the assumptions on B one need only consider singular cubes of X whose faces of dimension $\leqq r$ are contained in F. *E. H. Spanier* (Chicago, Ill.).

Whitehead, George W. On the homology suspension. Ann. of Math. (2) **62** (1955), 254–268.

Soient B un espace connexe et simplement connexe, $b_0 \in B$, E l'espace des chemins de B, E_0 (resp. E_1) l'espace des chemins d'origine b_0 (resp. d'extrémité b_0), $F=E_0 \cap E_1$ l'espace des lacets au point b_0, $X=E_0 \cup E_1$. Soient $B \vee B$ le joint basé en b_0, et $F \vee F$ le joint basé au point e (élément neutre de l'espace des lacets); soit $p' : E \to B \times B$ l'application qui à chaque chemin associe son origine et son extrémité. L'auteur établit le diagramme commutatif

$$\begin{array}{ccccccccc} & & & & H_{q+1}(B\times B, B\vee B) & & & & \\ & & & & \uparrow s_1 & & & & \\ H_q(F) & \xrightarrow{\sigma} & H_{q+1}(B) & \xrightarrow{\tau} & H_{q+1}(E, X) & \xrightarrow{\varrho} & H_{q-1}(F) & \xrightarrow{\sigma} & H_q(B) \\ & & & & \uparrow \beta & \nearrow \lambda & & & \\ & & & & H_{q-1}(F\times F, F\vee F) & & & & \end{array}$$

où la suite horizontale est exacte. Les groupes d'homologie sont pris avec coefficients dans un Γ-module, M, Γ désignant un anneau d'intégrité principal; σ désigne la suspension; τ se déduit de l'isomorphisme naturel $H_{q+1}(E) \approx H_{q+1}(B)$; ϱ se déduit d'un isomorphisme $H_{q-1}(F) \approx H_q(X)$; p_q est défini par l'application fibrée p', de fibre F; d_2 est défini par l'application diagonale $B \to B \times B$; λ se déduit de la multiplication dans l'espace des lacets F; la définition de β est plus compliquée.

L'auteur montre que si $H_q(B; \Gamma)=0$ pour $0<p \leqq n$, alors p_q et β sont des isomorphismes (sur) pour $q \leqq 3n$. On a un diagramme et un résultat analogues en cohomologie.

Application: on sait que les opérations cohomologiques (partout définies) $H^n(K; \pi) \to H^q(K; G)$ correspondent biunivoquement aux éléments $u \in H^q(\pi, n; G)$ de la cohomologie d'Eilenberg-MacLane. Si

$$u=\sigma^* u', \quad u' \in H_q^{+1}(\pi, n+1; G),$$

alors l'opération cohomologique définie par u est additive [Eilenberg et MacLane, Ann. of Math. (2) **60** (1954), 513–557, voir th. 16.2; MR **16**, 392]. L'auteur démontre la réciproque pour $q<3n$.

Erratum: dans les diagrammes des lemmes 2.3 et 2.4, reverser le sens de toutes les flèches verticales; dans la Fig. 1, renverser le sens de la flèche β; dans le bas de la Fig. 2, remplacer $H^{q-1}(X, e)$ par $H^{q+1}(X, e)$. Page 265, avant-dernière ligne, lire $u \in H_p(F, e)$; p. 266, ligne 14, lire deux fois H^{q-1} au lieu de H_{q-1}. *H. Cartan.*

Deheuvels, René. Notion de suite exacte de faisceaux localement triviale. C. R. Acad. Sci. Paris **240** (1955), 1183–1185.

Let $0 \to A \to B \to C \to 0$ be an exact sequence of sheaves (faisceaux) of abelian groups over the space X. For such a sequence the notions of being trivial, locally trivial, completely trivial, and completely locally trivial over X are introduced. For F and G two sheaves of abelian groups over X, the sheaf $(F; G)$ is defined by means of homomorphisms. This is somehow dual to $F \circ G$, the tensor product of F and G over X. Theorems of the following type are stated. Let $0 \to A \to B \to C \to 0$ and F be as before. The sequence

$$0 \to (F; A) \to (F; B) \to (F; C) \to 0$$

is exact if and only if $0 \to A \to B \to C \to 0$ is locally trivial over X. *J. Adem* (Mexico, D.F.).

Deheuvels, René. Classe caractéristique d'une application continue. C. R. Acad. Sci. Paris **240** (1955), 1298–1300.

Using some of the concepts of the paper reviewed above, the author defines under some general hypotheses the characteristic class $\gamma(\xi)$ of a continuous transformation $\xi : X \to Y$. The $\gamma(\xi)=\sum_{n>0} \gamma^n(\xi)$ contains cohomology classes of all dimensions >0, and not only in the critical dimension as in the classic case. It can be used to obtain the differentials δ_r of the spectral sequence of ξ considered elsewhere by the author [same C.R. **238** (1954), 1286–1288; MR **15**, 890; and the paper reviewed below]. Also, if ξ is a fibering and the fibre is homologically a sphere, $\gamma(\xi)$ becomes the classic characteristic class. The author promises some applications. *J. Adem* (Mexico, D.F.).

Deheuvels, René. Invariants d'une application continue. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 90–93.

The author develops a theory where he obtains a cohomology spectral sequence for an arbitrary map of one space into another. This is done as an extension of work of Leray [J. Math. Pures Appl. (9) **29** (1950), 1–80, 81–139; MR **12**, 272], the work of Spanier [Proc. Internat. Congress, Math., Cambridge, 1950, v. 2, Amer. Math. Soc. Providence, R.I., 1952, pp. 390–396; MR **13**, 486] and the work of Massey [Ann. of Math. (2) **56** (1952), 363–396; **57** (1953), 248–286; MR **14**, 672, 1111]. *J. C. Moore.*

Bott, Raoul. On manifolds all of whose geodesics are closed. Ann. of Math. (2) **60** (1954), 375–382.

Die Arbeit handelt von Riemannschen Mannigfaltigkeiten M der Klasse C^3, dim $M>1$, in denen es einen solchen Punkt p gibt, dass alle von ihm ausgehenden geodätischen Linien einfach geschlossen und von der gleichen Länge w sind. Man sieht leicht, dass eine solche M kompakt ist. Der "Index" λ einer der genannten geodätischen Linien g ist die Anzahl (mit Zählung von Vielfachheiten) der zu p konjugierten Punkte q auf g, für die der Bogen pq kürzer als w ist. Im Laufe der Arbeit wird gezeigt (Folgerung aus Theorem 4.1), aber nicht als Theorem formuliert, dass λ nur von p, nicht von g abhängt. Der Hauptsatz lautet: "Wenn $\lambda=0$ ist, so hat die Fundamentalgruppe von M die Ordnung 2, und der universelle Überlagerungsraum ist eine Homologiesphäre. Wenn $\lambda>0$ ist, so ist M einfach zusammenhängend, und der ganzzahlige Cohomologiering von M ist der Ring der Polynome in einer Grösse x, die eine Relation $x^n=0$ erfüllt und die Dimension $\lambda+1$ hat."

Die Voraussetzung, dass die geodätischen Linien durch p gleich lang sind, ist für die Gültigkeit des Satzes wesentlich. Die Frage, ob ihre Einfachheit (Doppelpunktfreiheit) wesentlich ist, bleibt offen. Der Beweis benutzt die Morsesche Theorie der kritischen Punkte und die Leraysche Theorie der Spektralfolgen. Die Hauptidee ist die Bestimmung der Homologiegruppen des Raumes der Schliefen („loops") in M. Der Verfasser teilt mit, dass H. Samelson mit einer direkteren Methode (Abbildung des projektiven Raumes in M) ähnliche Resultate erhalten hat. *H. Hopf* (Stanford, Calif.).

Wu, Wen-Tsün. On Pontrjagin classes. I. Sci. Sinica 3 (1954), 353–367.

This paper presents two different intrinsic definitions of the Pontryagin classes attached to a differentiable manifold [for the first definition see also S. S. Chern, Topics in differential geometry (mimeographed), Inst. Advanced Study, Princeton, 1951; and A. Borel and J. P. Serre, Amer. J. Math. **75** (1953), 409–448; MR **15**, 338; the second, as noted in the paper, was also given by G. Hirsch, C. R. Acad. Sci. Paris **230** 46–48 (1950),; MR **11**, 379]. In §§ 1, 2 real and complex Grassmann manifolds are investigated; the images of the Chern classes, under the cohomology map associated with the imbedding of the real into the complex Grassmann manifold, are determined in terms of the Pontryagin and Stiefel-Whitney classes, mod odd primes, mod 2, rational and also integral; in the latter formula there appear, besides the integral Pontryagin classes, certain images of Stiefel-Whitney elements under the Bockstein map. In § 3 the integral Pontryagin classes of dimensions $4k$ of a real vector-space bundle are then defined in terms of the Chern classes of the associated complex vector space bundle and the Stiefel-Whitney classes of the original bundle. For a second intrinsic definition, one associates with a unitary bundle in a natural fashion a bundle, whose fibre is a complex projective space, and constructs, over this bundle as base, a complex line bundle. The Chern classes of the original bundles can then be defined in terms of the powers of the 2-dimensional characteristic class of the latter bundle. The Pontryagin classes are again defined by extending a real vector-space bundle to a complex vector-space bundle. *H. Samelson* (Ann Arbor, Mich.).

Wu, Wen-Tsün. On Pontrjagin classes. I. Acta Math. Sinica 3 (1953), 291–315. (Chinese. English summary)

An English version of this paper is reviewed above.

See also: Hirzebruch, p. 535; Roth, p. 536.

GEOMETRY

van Beylen, E. On the largest number of parts into which a convex polygon can be divided by its diagonals. Nieuw Tijdschr. Wisk. **43** (1955/56), 133–135. (Dutch)

Brun, Viggo. On the problem of partitioning the circle so as to visualize Leibniz' formula for π. Nordisk Mat. Tidskr. 3 (1955), 159–166, 183.

★**Suter, Herbert. Complexes quadratiques homofocaux, complexes de tangentes et complexes à tétraèdres polaires.** Thèse, Ecole Polytechnique Fédérale, Zurich, 1955. Imprimerie Centrale S. A., Neuchatel, 1955. 48 pp.

This is a vectorial treatment of some families of quadratic complexes in three dimensions, with applications to statics. The author obtains the equation for the complex of tangents to a quadric, and for the Painvin complex: the class of lines from which the tangent planes to a given quadric are perpendicular.

H. S. M. Coxeter (Toronto, Ont.).

Neumann, Maria. Sur les invariants projectifs de la forme du IV-ième degré. Gaz. Mat. Fiz. Ser. A. 7 (1955), 581–583. (Romanian. Russian and French summaries)

Škorpík, Vladimír. On hyperosculating conic sections. Časopis Pěst. Mat. **80** (1955), 232–241. (Czech)

Sispánov, Sergio. An analytic method for tracing quadratics and cubics. Rev. Un. Mat. Argentina **16** (1955), 129–150. (Spanish)

De Sloovere, H. La tétragonométrie euclidienne déduite de la trigonométrie sphérique. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 1130–1131.

Zăgănescu, M. Représentation du procès polytropique en coordonnées homogènes. Gaz. Mat. Fiz. Ser. A. 7 (1955), 553–555. (Romanian. Russian and French summaries)

Gran Olsson, R. Über Porenvolumen und Porenziffer in der Erdbaumechanik und die verschieden dichte Packung von Kugeln. Norske Vid. Selsk. Forh., Trondheim 28 (1955), 96–99.

The known ratios of pore space to solid for three different regular packings of equal spheres (loose, medium and dense) are listed. They are compared with observed values of this ratio, of water content and specific gravities for various soils. *M. E. Wise* (Cardiff).

Castrucci, Benedito. Foundations of N-dimensional finite projective geometry. Bol. Soc. Mat. São Paulo **7** (1952), i–ii, 1–83 (1954). (Portuguese)

This is an expository thesis defended in 1951 at the University of São Paolo. It is competently written and might be of interest to a beginning graduate student. The topic is analytic N-dimensional projective geometry over an arbitrary finite field, with considerable emphasis on enumeration. In so far as the reviewer can judge in 1956, nothing in the paper would have seemed new to him in 1951 when it was written. The bibliography of 46 items is somewhat misleading in that it contains many papers (for example, seven papers of R. Moufang) which are essentially irrelevant to the topic at hand.

R. H. Bruck (Madison, Wis.).

Artzy, R. Loops and generally situated 4-webs. Technion. Israel Inst. Tech. Sci. Publ. **6** (1954/5), 5–13. (Hebrew summary)

Continuing his investigation of generally situated 4-webs [Math. Ann. **126** (1953), 336–342; MR **15**, 346] the author gives six postulates characterizing a type of semiloop which can be used to coordinatize such a 4-web. Such a semiloop S has a zero, u, and each x of S has a "negative" x' defined by $x+x'=u$. The author produces the web configuration corresponding to the additional axiom: $x+(y+x')=y$ for all x, y in S. The seven axioms are satisfied by an infinite set of non-isomorphic loops defined elsewhere by the author [Proc. Amer. Math. Soc. **6** (1955), 448–453; MR **16**, 1083] and also by the additive loop of a planar neofield supplied by H. Naumann. Historical remark: The author credits the concept of Moufang loops to G. Bol [Math. Ann. **114** (1937), 414–431]. These loops are in fact the quasigroups of R. Moufang [ibid. **110** (1934), 416–430]. *R. H. Bruck.*

Havel, Václav. Harmonical quadruplet in Moufang plane. Czechoslovak Math. J. **5(80)** (1955), 76–82. (Russian summary)

This paper essentially contains nothing new and seems mainly intended to list, for use in the paper reviewed below, the explicit relation between the coordinates of four points of the x-axis, which means that they form a harmonic quadruplet. Coordinates are taken from an alternative division ring. *R. H. Bruck.*

Havel, Václav. On the theorem of Staudt in Moufang plane. Czechoslovak Math. J. **5(80)** (1955), 83–90. (Russian summary)

The author considers projective planes which satisfy the axiom of the complete quadrilateral and hence can be coordinatized by alternative division rings of characteristic not 2. A Staudt projectivity of line p of plane π upon line p' of plane π' is a one-to-one mapping of the points of p upon the points of p' which preserves the harmonic relation. The author proves that such a mapping is the product of a projectivity of p upon p' and a mapping of p' induced by an automorphism or anti-automorphism of a suitable coordinate ring of π'. To do so he uses the relation of the paper reviewed above to reduce the problem to one concerning Jordan homomorphisms of alternative rings. He then has available known results on Jordan homomorphisms and semi-homomorphisms of associative rings which he is able to carry over to alternative rings by minor additions to proofs given by Hua [Uspehi Mat. Nauk (N.S.) **8** (1953), 3(55), no. 143–148; MR **15**, 392]. As the author states, the theorem is known for Desarguesian planes. *R. H. Bruck* (Madison, Wis.).

Klingenberg, Wilhelm. Desarguessche Ebenen mit Nachbarelementen. Abh. Math. Sem. Univ. Hamburg **20** (1955), 97–111.

The author carries over to planes with neighbour elements [Klingenberg, Math. Z. **60** (1954), 384–406; MR **16**, 507] results of Artin [Rep. Math. Colloq. **2** (1940), 15–20; MR **3**, 179]. Let π be a system consisting of points, lines and an incidence relation such that two distinct points (lines) are both incident with at least one line (point). Two distinct points (lines) are called neighbours if they are both incident with more than one line (point). If θ is the operation of identifying neighbour points and neighbour lines, then π is called a projective plane with neighbours (or a Hjemslev plane) provided $\pi\theta$ is a projective plane in the usual sense. Let π be such a plane. Distinguish a line L of π and call a point P (line M) of π proper if $P\theta$ is not on $L\theta$ ($M\theta \neq L\theta$). Call a collineation τ of π a translation if it preserves "properness". Call any proper line left fixed by τ a trace of τ. The author proves that $\pi\theta$ is Desarguesian if and only if the following axioms hold in π for some choice of L: (1) The translations form a group G which is simply transitive on the proper points. (2) If two translations τ, τ' have a common trace there exists a trace-preserving endomorphism of G which maps τ on τ'. More precisely, the two conditions imply that π can be assigned coordinates from a simple Hjemslev ring H (the ring of trace-preserving endomorphisms of G). Here H is an associative ring with identity element whose two-sided zero-divisors form an ideal N such that H/N is a division ring. The author remarks that Desarguesian planes with neighbour elements are precisely those imbeddable in (analogously defined) higher-dimensional spaces with neighbour elements and mentions a forthcoming paper.

R. H. Bruck (Madison, Wis.).

Skornyakov, L. A. Metrization of a projective plane in connection with a given system of curves. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 471–482. (Russian)

A two-dimensional G-space [for this and the following see the reviewer's The geometry of geodesics, Academic press, New York, 1955] in which the geodesic through two points is unique is either 1) homeomorphic to the plane and all geodesics are isometric to the real axis or 2) homeomorphic to the projective plane P^2 and all geodesics are isometric to the same circle [see loc. cit. Theorems (10.7) and (31.2)]. The reviewer determined all systems of curves which can serve as geodesics in case 1) [loc.cit. Theorem (11.2)] but left the corresponding problem for case 2) open [loc.cit. page 403, Problem (9)]. This problem is settled: On P^2 let a system Σ of curves be given such that each curve in Σ is homeomorphic to a circle and two given distinct points of P^2 lie on exactly one curve of Σ. Then P^2 can be metrized as a G-space such that the curves in Σ become the geodesics. Whereas the reviewer had thought that this problem would require essentially different methods from those used by him for case 1), the author actually uses an ingenious modification of essentially the same method.

H. Busemann.

See also: Conseth, p. 448; Hughes, p. 451; Heppes, p. 523.

Convex Domains, Extremal Problems, Integral Geometry, Distance Geometry

Ghika, Al. Séparation des ensembles convexes dans les espaces lignés non vectoriels. Acad. Repub. Pop. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 287–296. (Romanian. Russian and French summaries)

The author uses as axioms a list of properties of the family of all convex sets in a linear space. He shows that in a space with a family of subsets satisfying these properties many other properties of convex sets can be proved in the family; in particular, the Eidelheit separation theorem can be proved by this method.

M. M. Day (Urbana, Ill.).

Coxeter, H. S. M. Arrangements of equal spheres in non-Euclidean spaces. Acta Math. Acad. Sci. Hungar. **5** (1954), 263–274. (Russian summary)

The author considers regular arrangements of spheres in non-Euclidean geometry giving close packings and thin coverings. In three-dimensional elliptic space the regular honeycomb $\{5, 3, 3\}$ consists of 60 dodecahedra $\{5, 3\}$ filling the elliptic space whose in- and circumspheres provide a packing of density 0.774 and a covering of density 1.439, which compare with the conjectured closest and thinnest densities of 0.740 and 1.464 in Euclidean 3-space. In hyperbolic 3-space no superior arrangements of finite spheres are known, but horospheres provide a packing of density 0.853 and a covering of density 1.280. This compares with Fejes Tóth's result that in hyperbolic 3-space no arrangements of finite circles are as good as the best arrangement of horocycles. [Acta Math. Acad. Sci. Hungar. **4** (1953), 103–110, 111–114; MR **15**, 341]. Finally, in hyperbolic 4-space the regular honeycomb $\{5, 3, 3, 3\}$ yields arrangements of finite spheres having packing density 0.691 and covering density 1.558, which compare with the best known densities of 0.617 and 1.766 in Euclidean 4-space.

R. A. Rankin (Glasgow).

Rankin, R. A. The closest packing of spherical caps in n dimensions. Proc. Glasgow Math. Assoc. **2** (1955), 139–144.

Consider a regular polytope inscribed in an $(n-1)$-sphere S_n in Euclidean n-space. Following the notation of the reviewer's "Regular polytopes" [Methuen, London, 1948; MR **10**, 261], let 2ϕ denote the angle subtended at the center by an edge. The N_0 vertices are the centers of N_0 non-overlapping spherical caps of angular radius ϕ.

When the polytope is a k-dimensional regular simplex α_k $(1\leqq k\leqq n)$, we have $2\phi=\pi-$ arc sec k and $N_0=k+1$ [op. cit., p. 295]. The author proves that this arrangement is the closest packing of $k+1$ spherical caps $(1\leqq k\leqq n)$. In other words, letting $N(\phi)$ denote the maximum number of spherical caps of angular radius ϕ that can be placed on an S_n without overlapping, we have

$$N(\phi)=k+1 \text{ for } \pi-\text{arc sec } (k+1)<2\phi\leqq\pi-\text{ arc sec } k,$$

$1\leqq k<n$, and $N(\phi)=n+1$ for $2\phi=\pi-$ arc sec n; that is,

$$N(\phi)=[\sec(\pi-2\phi)]+1 \text{ for } \pi-\text{ arc sec } n\leqq 2\phi\leqq\pi.$$

In the case of the cross-polytrope β_n, whose vertices are the permutations of $(\pm 1, 0, 0, \cdots, 0)$, we have $\phi=\frac{1}{4}\pi$ and $N_0=2n$. The author proves the remarkable fact (anticipated by Hajós and Davenport) that, if $n+2\leqq N\leqq 2n$, N non-overlapping caps of radius $\frac{1}{4}\pi$ have their centers at N of the $2n$ vertices of a β_n. It follows that $N(\phi)$ cannot take a value between $n+1$ and $2n$: $N(\phi)=n+1$ for $\frac{1}{3}\pi<2\phi\leqq\pi-$ arc sec n, but $N(\frac{1}{3}\pi)=2n$.

For acute values of 2ϕ, he uses the density method of Blichfeldt [Math. Ann. **101** (1929), 605–608] to obtain an upper bound for $N(\phi)$. This is a complicated expression $N^*(\phi)$ involving a trigonometric integral. If ϕ remains small while n increases, $N^*(\phi)$ is asymptotically

$$(\tfrac{1}{2}\pi n^3\cos 2\phi)^{\frac{1}{2}}/(2^{\frac{1}{2}}\sin\phi)^{n-1}.$$

For instance, when $n=4$, we have $N(\pi/10)=120$ [p. 266 of the paper reviewed above] whereas, in terms of $\tau=\frac{1}{2}(\sqrt{5}+1)$,

$$N^*(\pi/10)=\pi/\{(\tfrac{1}{3}+\tfrac{1}{3}\tau^3)\tau^{-\frac{1}{2}}-\tau^2\arccos(\tfrac{1}{2}\tau)^{\frac{1}{2}}\},$$

which is about 271. The asymptotic formula yields $8(2\pi)^{\frac{1}{2}}\tau^{7/4}$, which is about 108.

The author points out that $N(\pi/6)$ is an upper bound for the number of times that a positive definite quadratic form in n real variables attains its minimum (for integral values of the variables not all zero). For instance, when $n=8$, $[N^*(\pi/6)]=309$, whereas the extreme form

$$x_1^2-x_1x_2+x_2^2-x_2x_3+\cdots-x_6x_7+x_7^2-x_7x_8+x_8^2$$

attains its minimum 240 times, namely at the 240 vertices of the uniform polytope 4_{21} [Regular polytopes, p. 204].

H. S. M. Coxeter (Toronto, Ont.).

Rankin, R. A. On packings of spheres in Hilbert space. Proc. Glasgow Math. Assoc. **2** (1955), 145–146.

Consider, in Euclidean n-space $(n\geqq k)$, a cluster of $k+1$ $(n-1)$-spheres $S_n(a)$ of radius a, each touching all the others. Their centres are the $k+1$ vertices of a regular simplex α_k of edge $2a$ and circum-radius $R=a\,(2k/(k+1))^{\frac{1}{2}}$ [Coxeter, Regular polytopes, Methuen, London, 1948, p. 158; MR **10**, 261]. Since the spheres touch one another, this cluster is the closest packing of "solid" $S_n(a)$'s within an $S_n(R+a)$. Setting $R+a=1$, so that

$$\frac{1}{a}=1+\left(\frac{2k}{k+1}\right)^{\frac{1}{2}},\qquad k+1=\frac{2a^2}{a^2+2a-1},$$

it follows that, if $1\leqq a^{-1}\leqq 1+(2n/(n+1))^{\frac{1}{2}}$, the maximum number of $S_n(a)$'s that may be placed within an $S_n(1)$ without overlapping is

$$\left[\frac{2a^2}{a^2+2a-1}\right].$$

Since the angle subtended by an edge of α_k at its center is 2ϕ, where $\sin\phi=a/R$ and $\sec 2\phi=k$, this result is equivalent to Theorem 1 of the paper reviewed above. He now applies it to the limiting case of Hilbert space, where we can draw any number of spheres touching one another, and where the only regular polytope not lying in a finite-dimensional subspace is $\beta_\infty=\alpha_\infty=\{3, 3, \cdots\}$ with vertices $(\pm R, 0, 0, \cdots)$, $(0, \pm R, 0, \cdots)$, $\cdots$.

H. S. M. Coxeter.

Heppes, A. Über mehrfache Kreislagerungen. Elem. Math. **10** (1955), 125–127.

A pattern of equal circles in the Euclidean plane is called a k-fold packing if every point is inside at most k of the circles. The density of the packing is defined as in the classical case when $k=1$ [L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Springer, Berlin, 1953, Ch. III; MR **15**, 248]. The upper bound of the density for all possible k-fold packings is denoted by d_k; for instance, $d_1=\pi/\sqrt{12}=.9069\cdots$. Clearly $kd_1\leqq d_k<k$. The author shows that, for any $k\geqq 2$, $d_k>kd_1$;

in fact,

$$\frac{d_k}{k} \geqq \frac{8\pi}{7\sqrt{15}} = .9270\cdots.$$

For this purpose he considers the lattice whose elementary cell is a rhombus $ABCD$ of side 4 with its longer diagonal $AC=7$. Circles of radius 2 around the lattice points do not form a 1-fold packing, as they overlap. But successive unit translations along AC provide k-fold packings for $k=2, 3, \cdots, 7$. *H. S. M. Coxeter* (Toronto, Ont.).

Süss, Wilhelm. Über die kleinste Kugel, die jede Figur gegebenen Durchmessers einschliesst. Abh. Math. Sem. Univ. Hamburg **20** (1955), 115–116.

The theorem of Gale that an equilateral n-simplex whose inscribed sphere has diameter 1 will cover any n-dimensional set of diameter 1 [Proc. Amer. Math. Soc. **4** (1953) 222–225; MR **14**, 787] is established by a very simple argument, and the result is applied to yield an easy elementary proof of the dual (Jung) theorem that every set of diameter 1 in n-space can be inscribed in an n-sphere of radius $r \leqq [n/2(n+1)]^{\frac{1}{2}}$. *L. M. Blumenthal.*

Fejes Tóth, László. Remarks on polygon theorems of Dowker. Mat. Lapok **6** (1955), 176–179. (Hungarian. Russian and German summaries)

Vereinfachter Beweis von Sätze von Dowker. [Bull. Amer. Math. Soc. **50** (1944), 120–122; MR **5**, 153].

Author's summary.

Hsu, L. C. A combinatorial proof of an inequality due to Hua. Math. Student **23** (1955), 97–100.

A combinatorial proof is given of the following theorem that the author ascribes to L. K. Hua (no reference supplied): If $a_1, \cdots, a_k, b_1, \cdots, b_k$ are two metric k-tuples and T is a positive constant such that

$$\sum_{i,j=1}^{k} a_i a_j \leqq T, \quad \sum_{i,j=1}^{k} b_i b_j \leqq T, \quad a_i a_j + b_i b_j > 0,$$

then the smallest of the non-zero distances $a_i a_j$, $b_i b_j$ is at most $3T/2k(k-1)$. *L. M. Blumenthal* (Columbia, Mo.).

Petty, C. M., and Waterman, D. An extremal theorem for n-simplexes. Monatsh. Math. **59** (1955), 320–322.

The author shows that the sum of the distances from an arbitrary point P to the vertices of an n-dimensional simplex of volume Δ is at least

$$(n+1)^{(n-1)/2n} n^{\frac{1}{2}} (n!)^{1/n} \Delta^{1/n}.$$

This value is attained only if the simplex is regular and P is its centroid. *W. T. Tutte* (Toronto, Ont.).

Hadwiger, H. Kleine Studie zur kombinatorischen Geometrie der Sphäre. Nagoya Math. J. **8** (1955), 45–48.

The author considers points P, Q, etc., on the n-sphere S_n of unit radius. By $d(P, Q)$ is meant the spherical distance of P and Q; then $d(P, Q) \leqq \pi$. For fixed P, the set of all points Q satisfying $d(P, Q) \leqq \pi/2$ is called a half-space and P its pole. A convex set on S_n is defined to be the intersection of a collection of half-spaces. The set of poles of those half-spaces containing a convex set K is also a convex set, the polar set of K. For fixed P, the set of all points Q satisfying $d(P, Q) \leqq s$, where $0 \leqq s \leqq \pi$, is a spherical ball of radius s and center P. If K is a convex set, its inner radius r and its outer radius R are the maxima and minima respectively of radii of spherical balls contained in and containing K, respectively. The author proves the following theorem A and its polar theorem B:

A. To any collection of convex sets on S_n each of which satisfies the condition $\sin R \leqq 1/(n+1)$ there corresponds a set of $n+2$ half-spaces with the property that each convex set of the collection is contained in at least one of the half-spaces.

B. To any collection of convex sets on S_n each one of which satisfies the condition $\cos r \leqq 1/(n+1)$ there corresponds a set of $n+2$ points with the property that each convex set of the collection contains at least one point of the set.

The theorems are sharp in that the number $(n+2)$ of half-spaces (or points) cannot be replaced by any smaller number, and the value $1/(n+1)$ cannot be replaced by any larger value. The author also states the following problem: Suppose that in Theorem A, it is assumed that the convex sets of the collection are mutually disjoint. Then is it possible to dispense altogether with the condition that $\sin R \leqq 1/(n+1)$? For $n=1$, this is trivially provable, but the question is open for $n>1$.

J. W. Green (Los Angeles, Calif.).

Tavora, Elysiario. The vector area concept and the determination of the relative reticular density of lattice planes. An. Acad. Brasil. Ci. **27** (1955), 137–140.

Nasu, Yasuo. On the normality in Minkowskian spaces. Kumamoto J. Sci. Ser. A. **2** (1954), no. 1, 11–17.

The concept of normality in Minkowski spaces in the sense of the reviewer [Comment. Math. Helv. **24** (1950), 156–187; MR **12**, 527] is treated analytically under certain differentiability hypotheses. This was done previously by Barthel [Math. Z. **58** (1953), 358–375; MR **15**, 149], and the author uses Barthel's methods, but completes his results by discussing a case overlooked by Barthel.

The author also proves briefly a theorem of the reviewer [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 27–31; MR **10**, 395]. But it seems to the reviewer that the author's proof is simple only because it does not justify the principal step which consists, roughly, in an interchange of convexity and integration. *H. Busemann.*

Beckert, Herbert. Bemerkungen zu meiner Arbeit "Über die Verbiegung von Flächenstücken positiver Krümmung und einige Bemerkungen zum Verhalten der Lösungen partieller Differentialgleichungen im Übergangsgebiet". Math. Nachr. **14** (1955), 21–23.

A proof that a piece of a convex surface with curvature $<4\pi$ is not rigid. Stronger results are proved in Pogorelov's "Deformation of convex surfaces" [Gostehizdat, Moscow-Leningrad, 1951; MR **14**, 400]. *H. Busemann.*

Haantjes, J., and Nottrot, R. Distance geometry. Directions in metric spaces. Torsion. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 405–410.

The authors consider a complete locally metrically convex metric space M with the property that for each accumulation point p and each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $\det |\cos p' : p_i, p_j| > -\varepsilon$ $(i, j = 1, 2, 3)$ whenever $pp_i < \delta$ $(i = 1, 2, 3)$, where $p' : p_i, p_j$ denotes the angle at the vertex p' of the planar triangle whose vertices p', p_i', p_j' are congruent, respectively, to p, p_i, p_j $(i, j = 1, 2, 3)$. [This implies that the limit of the "trihedral angle" formed at p by p_1, p_2, p_3 as the latter three points approach p is isogonally imbeddable in euclidean threespace; that is, that M is infinitesimally trihedral at each point p. It

follows that Riemannian spaces belong to the class of spaces treated in this paper.] A subset C of M is called a T-set at p_0, an accumulation point of M, provided $\lim_{q,r\to p_0} \cos p_0{:}q, r=1$ $(q, r \in C)$, and the angle α_{12} $(0\leq\alpha_{12}\leq\pi)$ between two T-sets C_1, C_2 at p_0 is defined by $\cos\alpha_{12}=\lim_{q,r\to p_0} \cos p_0{:}q, r$ $(q \in C_1, r \in C_2)$. It is proved that α_{12} exists for any two T-sets at p_0, and if C_1, C_2, C_3 are T-sets at p_0, then the three angles they determine satisfy the triangle inequality and have a sum at most 2π. An equivalence relation is defined in the class of T-sets at p_0 by putting $C_1\sim C_2$ provided $\alpha_{12}=0$. This permits defining a direction D of M at p_0 as a class of equivalent T-sets. Any member of the class is said to have direction D at p_0, and the angle between two directions at p_0 is taken as the angle between any two members of the respective equivalence classes defining the directions. It follows that the set of directions at p_0 forms a metric space N_1, if the distance of two directions is defined as the angle between them. Two T-sets have opposite directions at p_0 if the angle between them is π. Identifying opposite directions d, d^* at p_0, and defining $\min[\sphericalangle(d_1, d_2), \sphericalangle(d_1, d_2^*)]$ as the distance of two pairs (d_1, d_1^*), (d_2, d_2^*) of opposite directions, gives another metric space N_2, the space of tangents at p_0. An arc $C[p(t), \alpha\leq t\leq\beta]$ has a tangent at $p_0=p(t_0)$ provided the two sub-arcs $p(t)$, $\alpha\leq t\leq t_0$, and $p(t)$, $t_0\leq t\leq\beta$, have opposite directions at p_0. Now let C be an arc with a tangent at $p_0=p(t_0)$. The directions of the metric segments $S_t=S(p_0, p(t))$ joining p_0 and $p(t)$, together with the direction d_0 of C at p_0 form a subspace C' of the metric space N_2 of tangents at p_0, and d_0 is an accumulation point of C'. Guided by the relation in euclidean space between the torsion of a curve and the curvature of the spherical indicatrix of its tangents, the authors define the torsion τ of C at p_0 as $(3/4)\varkappa\varrho$, where $\varkappa$ is the metric curvature of C at p_0 and ϱ is the metric curvature at d_0 of C', provided the latter exists. It is stated that in a following paper this notion of torsion will be compared with the notion introduced by the reviewer [Theory and applications of distance geometry, Oxford, 1953; MR **14**, 1009]. *L. M. Blumenthal*

See also: Remez, p. 477; Aleksandrov, p. 493; Nikaidô and Isoda, p. 506; Iséki and Kasahara, p. 510; Smythe, p. 517; Nasu, p. 527.

Differential Geometry

Bouligand, Georges. Types de courbes planes et de surfaces. C. R. Acad. Sci. Paris **241** (1955), 145–148.

Bouligand, Georges. Sur un type de courbes et de surfaces s'offrant à l'analyse linéaire. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 734–740.

Suppose the surfaces $x(u, v)$ and $x^*(u, v)$ have parallel tangent planes at corresponding points; x^* is assumed to be convex and symmetric with respect to the origin. Let h (h^*) be the support function and let M (M^*) [K (K^*)] denote the mean [Gaussian] curvature of x (of x^*). The straight lines $x+\lambda x^*$ envelop two surfaces $x+\varrho_i x^*$ $(i=1, 2)$. The first note deals with surfaces x which satisfy $\varrho_1+\varrho_2=\text{const}\times h/h^*$ for a given x^*. In the second paper, the condition $h^{-1}M/K=\text{const}\times h^{*-1}M^*/K^*$ is imposed instead. If x^* is a sphere, either condition reduces to $M/K=\text{const}\times h$. This case as well as the corresponding classes of plane curves are discussed in both papers. (If the surface $x+x^*$ has the Gaussian curvature $\tilde{K}$, then $\varrho_1+\varrho_2=K^*(K^{-1}+K^{*-1}-\tilde{K}^{-1})$.) *P. Scherk.*

Arghiriade, E. Sur le contact de deux courbes gauches. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 735–739. (Romanian. Russian and French summaries)

This paper establishes the following relation between Halphen's principal plane p_1 of two space curves C, C' having contact of order one at a point O and the invariant of Smith-Mehmke k of their projections from a point V onto a plane: $(\alpha\alpha' p_1 p_k)=k$. Here α, α' are the osculating planes of C, C' in O and p_k is the plane through V and the common tangent in O. *R. Blum* (Saskatoon, Sask.).

Pinl, M. Explicite representations of real isotropic curves in pseudospherical three- and fourdimensional spaces. Rend. Mat. e Appl. (5) **14** (1955), 686–695.

Rahmat Ara and the author [J. Reine Angew. Math. **192** (1953), 204–209; MR **15**, 739] have given explicit representations of real isotropic curves in spaces with metrics a) $d\xi_1^2+d\xi_2^2-d\xi_3^2$, b) $d\xi_1^2+d\xi_2^2+d\xi_3^2-d\xi_4^2$, and c) $d\xi_1^2+d\xi_2^2-d\xi_3^2-d\xi_4^2$. Pseudostereographic projections will carry isotropic curves into isotropic curves, and the author now gives explicit representations of the corresponding real isotropic curves on pseudohyperspheres a') $x_1^2+x_2^2+x_3^2-x_4^2=1$, b') $x_1^2+x_2^2+x_3^2+x_4^2-x_5^2=1$, and c') $x_1^2+x_2^2+x_3^2-x_4^2-x_5^2=1$. Two real totally isotropic surfaces contained by c') are given, each curve on these surfaces being isotropic on the surface and on c'). *A. Schwartz* (New York, N.Y.).

Fog, David. A small problem from the theory of surfaces. Nordisk Mat. Tidskr. **3** (1955), 157–158, 183. (Danish. English summary)

It is shown that the function

$$z=y^2-2x^2y+\tfrac{1}{4}x^4=(y-\tfrac{1}{2}x^2)(y-\tfrac{3}{2}x^2)$$

does not have a minimum for $x=y=0$, in spite of the fact that for the corresponding surface, all the normal sections through the origin have a minimum at this point.

Author's summary.

Nevanlinna, Rolf. Die konformen Selbstabbildungen des euklidischen Raumes. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **19** (1954), 133–139 (1955).

This paper gives a new proof that for $n\geq 3$ the only conformal mappings of Euclidean n-dimensional space onto itself are of the forms

$$y(x)=\lambda UTx, \quad y(x)=\lambda URTx,$$

where T is a translation, U is a rotation, and R is an inversion. The scalar factor λ yields a homothetic transformation. Unlike the method of the classical proof, no use is made of the fundamental equations of the theory of surfaces. The notation employed is that of the differential calculus in vector spaces with metric arising from an inner product. The arguments apply without modification to Hilbert space. *A. E. Taylor* (Los Angeles, Calif.).

Şemin, F. Note sur les dérivées invariantes. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **19** (1954), 175–179 (1955). (Turkish summary)

La notion de derivée invariante dans la théorie des surfaces est déjà connue dans le cas où la surface est rapportée à ses lignes de courbure. L'A. étends cette notion dans le cas où la surface est rapportée à lignes coordonnées u, v quelconques. Si E, F, G sont les coefficients de la première forme fondamentale et si $f=f(u, v)$ est une fonction definie sur la surface, les derivées pre-

mières invariantes f_1, f_2 de f dans les directions u, v sont $f_1=f_u/\sqrt{E}$, $f_2=f_v/\sqrt{G}$. Les derivées secondes mixtes invariantes sont liées par la relation: $f_{12}+qf_1=f_{21}+\bar{q}f_2$. L'A. donne une signification géométrique des coefficients q et $\bar{q}$, l'expression de la courbure totale sous forme invariante et il retrouve certains résultats classiques relatifs aux réseaux de Tchebychef. *C. Longo* (Rome).

Gabadadze, N. A. On the application of complex and hypercomplex numbers to the theory of rectilinear congruences. Soobšč. Akad. Nauk Gruzin. SSR **15** (1954), 641–645. (Russian)

Following proposals of B. A. Rozenfel'd [Trudy Azerbaidžansk. Gos. Univ. **1** (1952), 33–68 (unavailable to the reviewer); One hundred and twenty-five years of the non-Euclidean geometry of Lobačevskiĭ 1826–1951, Gostehizdat, Moscow-Leningrad, 1952, pp. 151–166; MR **15**, 148] points of the absolute of the space ${}^1S_3$ for which $-(x^0)^2+(x^1)^2+(x^2)^2=0$ are represented by

$$z=(x^1+ix^2)/(x^0-x^3)$$

(this was already proposed by F. Klein); and points of the absolute of the space ${}^2S_3$ for which $-(x^0)^2+(x^1)^2-(x^2)^2+(x^3)^2=0$ are represented by $z=(x^1+\epsilon x^2)/(x^0-x^3)$, where $i^2=-1$ and $\epsilon^2=+1$. Since every line marks out two points on the absolute, a function $w=f(z)$ determines a rectilinear congruence. Expressions are given for the distribution parameter of lines of the congruence and other quantities, also for line congruences in projective P_3, where a related representation is possible with two-way matrices instead of complex or dual numbers. *D. J. Struik* (Cambridge, Mass.).

Godeaux, Lucien. Sur les quadriques de Lie des deux nappes d'une congruence W. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 1101–1103.

Godeaux, Lucien. Sur une homographie associée à une congruence W. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 870–874.

Su, Buchin. Asymptotic nets and harmonic congruences. Acta Math. Sinica **3** (1953), 167–176. (Chinese. English summary)

As a generalization of the results obtained by the author [On the surfaces whose Lie quadrics all touch a fixed plane, Science Reports, Chekiang Univ. **2** (1936), 53–61], the reviewer [Trans. Amer. Math. Soc. **70** (1951), 312–322; MR **13**, 276] and Bell [Proc. Amer. Math. Soc. **3** (1952), 300–302; MR **13**, 775], the following theorem is proved. Let P_1 and P_2 be two points respectively on the asymptotic tangents of a non-ruled proper surface (P) at an ordinary point P, and let the congruence (P_1P_2) be harmonic to the asymptotic net of the surface (P). If PP_3 is the line drawn from P to intersect the tangents, other than the common tangent P_1P_2, of the focal nets (P_1) and (P_2) at P_1 and P_2 respectively, then the two lines PP_3 and P_1P_2 are conjugate with respect to the Wilczynski quadric of the surface (P) at P.

C. C. Hsiung (Bethlehem, Pa.).

Barner, Martin. Doppelverhältnisscharen auf Regelflächen. Math. Z. **62** (1955), 50–93.

In a three-dimensional projective space S_3 if the two vectors

$$\mathfrak{q}_i(t)=\{q_0^{(i)}(t), q_1^{(i)}(t), q_2^{(i)}(t), q_3^{(i)}(t)\} \quad (i=1, 2),$$

are linearly independent, then $\mathfrak{q}_1(t)+s\mathfrak{q}_2(t)$, $s=$ const, generate the curves of a double ratio family F on a ruled surface, and conversely, every such family can be obtained in this way. This paper gives, with geometric discussions, an analytic projective theory of such families F of curves on a ruled surface in S_3. Every family F defines in the kinematic sense a projective motion, which leads to a hyperbolic motion in a plane. These two motions are studied and their relations to the geometry of families F are deduced. The characteristic curve of a one-parameter family of osculating quadrics associated with a family F is a quadrilateral. The relations between two double ratio families of curves on a ruled surface in S_3 are also discussed. *C. C. Hsiung* (Bethlehem, Pa.).

Barner, Martin. Zur projektiven Differentialgeometrie der Komplexflächen. III. Schiebungen als Abrollvorgänge. Singularitätenkurven auf Komplexflächen. Invariante Gebilde. Math. Ann. **129** (1955), 304–322.

As a continuation of two previous papers [Math. Ann. **126** (1953), 119–137, 418–446; MR **15**, 349, 467], the author studies "the momentarily fixed points" of a projective displacement. At any time these fixed points form two lines which generate a pair of ruled surfaces R_1 and R_2 with respect to a moving as well as a fixed system of reference. These two ruled surfaces R_1 and R_2 with respect to a moving system of reference are discussed for the three cases: I. R_1 and R_2 are distinct W-transforms of each other. II. R_1 and R_2 are coincident but not developable. III. R_1 and R_2 are coincident and developable.

C. C. Hsiung (Bethlehem, Pa.).

Vaona, Guido. Sulla deformazione proiettiva debole di uno strato di quadriche. Boll. Un. Mat. Ital. (3) **10** (1955), 337–348.

Una trasformazione puntuale T fra due spazi proiettivi S_3 ed $\bar{S}_3$ è una deformazione proiettiva debole di uno strato Σ di superficie di S_3 (ossia di una famiglia ∞^1 di superficie tale che per ogni punto di una data regione passi una ed una sola superficie della famiglia) in uno strato $\bar{\Sigma}$ di superficie di $\bar{S}_3$ quando la T subordina tra superficie corrispondenti dei due strati un'ordinaria applicabilità proiettiva e le giaciture piane tangenti alle superficie sono giaciture caratteristiche della T (ciascuna di tali giaciture contiene tre direzioni caratteristiche: se ciascuna direzione è caratteristica si ha un'ordinaria deformazione proiettiva). L'A. determina analiticamente e carratterizza geometricamente gli strati di quadriche proiettivamente deformabili in senso debole.

Considerate le congruenze costituite dalle generatrici di ciascuna schiera delle quadriche dello strato si hanno i seguenti casi: 1°) entrambe le due congruenze hanno falde focali degeneri e in tal caso si hache la deformazione proiettiva è ordinaria; 2°) una almeno delle due congruenze ha una falda focale non degenere ed una degenere: le quadriche di ciascun strato passano per una retta fissa e sono tangenti ad una superficie rigata lungo le sue generatrici; 3°) una almeno delle due congruenze ha due superficie focali non degeneri, distinte o coincidenti: gli strati appartengono ad una congruenza W a falde focali rigate.

Si ha inoltre che con le trasformazioni T che realizzano la deformazione debole fra due dati strati di quadriche dipendono da quattro funzioni arbitrarie di un parametro; ne segue: 1) un dato strato di quadriche proiettivamente deformabile in senso debole ammette deformazioni dipendenti da otto funzioni arbitrarie di una variabile;

2) le trasformazioni che sono deformazioni proiettive in senso debole di uno strato di quadriche dipendono da 12 funzioni arbitrarie di una variabile. *C. Longo* (Roma).

Bol, Gerrit. Zur projektiven Theorie der infinitesimalen Flächenverbiegung. Math. Z. **62** (1955), 1–16.

(*) $$P(u, v)+\varepsilon\overline{P}(u, v)$$

is called an infinitesimal deformation of a surface $P(u, v)$ in a three-dimensional Euclidean space, if the linear element of (*) is independent of the first order of ε. In this paper the author studies the relations between infinitesimal deformations of a surface and the W-congruence tangent to the surface, and in particular proves that an infinitesimal deformation of a surface without parabolic points corresponds to a unique solution of Fubini's fundamental equations of the surface and conversely. Infinitesimal deformations of special surfaces are also discussed.

C. C. Hsiung (Bethlehem, Pa.).

Terracini, Alessandro. Enti geometrici collegati con coppie di elementi curvilinei spaziali. Rend. Mat. e Appl. (5) **14** (1955), 439–454.

Suppose that in a three-dimensional projective space two elements of curves, C, C' have a contact of order $r\geqq 1$ with osculating planes τ, τ' and principal plane π at a common point O. Necessary and sufficient conditions for the projections of C, C' onto π from two distinct points to have a contact of order $>r$ are obtained for the three cases: I) $r=1$ and τ, τ' are distinct. II) $r=2$ and τ, τ' differ from π. III) $r=2$ and τ, τ' coincide with π. A special case of case III has been discussed in a previous paper by the author [Univ. e Politec. Torino. Rend. Sem. Mat. 12 (1953), 265–281; MR **15**, 826].

C. C. Hsiung.

Su, Buchin. The representable singularity of a curve in a projective space. J. Chinese Math. Soc. (N.S.) **1** (1951), 1–35. (Chinese summary)

In this paper the author studies the projective differential geometry of a curve in an n-dimensional projective space S_n at a singular point where the tangent of the curve has a μ-point contact with the curve, μ being an integer $\geqq 3$. $(n-1)(\mu-3)$ conditions of representability for such a singular point are established and canonical power-series expansions of the curve in the neighborhood of such a representable singularity are obtained by a geometric determination of a pyramid of reference.

C. C. Hsiung (Bethlehem, Pa.).

Mihăilescu, Tiberiu. Variétés non-holonomes du type Ţiţeica-Wilczinski. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **6** (1955), 175–192. (Romanian. Russian and French summaries.

It is shown that in a three-dimensional projective space the non-holonomic surfaces V_3^2, each of which has indeterminate focal curves of the congruence formed by its projective normals, exist and depend on four arbitrary functions of two arguments. The projective normals of each of these V_3^2 pass through a fixed point, so that these V_3^2 are the non-holonomic projective spheres. The non-holonomic surfaces V_3^2, each of which has indeterminate focal curves of the congruence C formed by the polar lines of its projective normals with respect to the fundamental polarity determined by the surface, form another class of surfaces depending also on four arbitrary functions of two arguments and the lines of the congruence C for each of these V_3^2 are in a fixed plane.

C. C. Hsiung.

Grotemeyer, Karl-Peter. Ein Satz aus der Differentialgeometrie der Eiflächen. Arch. Math. **6** (1955), 403–407.

Let C be a closed convex surface in E^3 of class C^4. If ∇ denotes the first Beltrami operator and N_1, N_2 are the principal curvatures, $H=\frac{1}{2}(N_1+N_2)$, $K=N_1N_2$, $S=H^2-K$, then for any positive integer n

$$\iint_C S^n\nabla(N_1, N_2)do\geqq 0$$

with equality only for the sphere. This implies Chern's result, that on a convex surface, except the sphere, one principal curvature cannot be a decreasing function of the other [Duke Math. J. **12** (1945), 279–290; MR **7**, 29]. However, Chern's result is already a special case of a theorem of A. D. Alexandrov [C.R. (Dokl.) Acad. Sci. URSS (N.S.) **22** (1939), 99–102].

The author derives his theorem from the following integral relation for a (not necessarily convex) surface G of class C^4 with a boundary B of class C^3:

$$\iint_G S^{n-1}\{2S^2K+(n+1)g^{\alpha\beta}\gamma_\alpha\gamma_\beta\}do=\oint_B S^n\gamma_\sigma u^\sigma ds+(n+1)\iint_G S^n\nabla(N_1, N_2)do,$$

where $\gamma_\sigma=\frac{1}{2}\varepsilon^{ik}B^t_k.B_{rtl\sigma}$ with B_{ik} denoting the tensor of the second fundamental form and ε_{ik} its discriminant tensor.

H. Busemann (Los Angeles, Calif.).

Grotemeyer, K. P. Zur eindeutigen Bestimmtheit konvexer Mützen. Arch. Math. **6** (1955), 454–461.

In contrast to a cap with a plane boundary, call "general cap" a surface A of positive curvature in E^3, which is of class C^3, is homeomorphic to a disk, has a boundary of class C^2 and is such that there exists a family of parallel lines which cut A at most once. The following is proved: Let A and A^* be isometric general caps whose boundaries have the spherical images S and S^*. If S and S^* are congruent, then A and A^* are congruent (reflection admitted). If both S and S^* are plane curves, then A and A^* are congruent. In the latter theorem the assumption that S^* be plane can be replaced by the requirement that its length does not exceed that of S.

H. Busemann.

Nasu, Yasuo. Remarks on spaces with non-positive curvature. Kumamoto J. Sci. Ser. A. **2** (1954), no. 1, 1–10.

The author defines non-positive curvature in a G-space [for this and other concept see the reviewer's book: The geometry of geodesics, Academic Press, New York, 1955, cited as G, in particular, Ch. V] as follows: Let $x_i(t_i)$, $\alpha_i\leqq t\leqq\beta_i$, $\alpha_i\leqq\beta_i$, $i=1, 2$, represent geodesic arcs and let $t_2=ct_1+d$ map $[\alpha_1, \beta_1]$ on $[\alpha_2, \beta_2]$. If $f(t)=x_1(t)x_2(ct+d)$, $\alpha_1\leqq t\leqq\beta_1$, then $\alpha_1\leqq\alpha'<t<\beta'\leqq\beta_1$ implies $f(t)\leqq\max(f(\alpha'), f(\beta'))$. By continuity this holds also for $\alpha_2=\beta_2$. F. P. Pedersen [Mat. Tidskr. B. **1952**, 66–89; MR **14**, 1015] defines non-positive curvature (called convexity of capsules in G to distinguish it from the author's stronger condition) as follows: in a neighborhood small enough for uniqueness of segments, consider two segments T_i $(i=1, 2)$ represented by $x_i(t_i)$, $\alpha_i\leqq t_i\leqq\beta_i$. If $\varphi(t)=x_1(t)T_2$, $\alpha_1\leqq t\leqq\beta_1$, then with the above notations $\varphi(t)\leqq\max(\varphi(\alpha'), \varphi(\beta'))$ with equality only for $\varphi(\alpha')=\varphi(\beta')$. That this condition on the equality may be omitted is implicitly proved by Pedersen and follows from (36.16) in G.

The author states that his condition is weaker than Pedersen's; actually it implies Pedersens' condition (but the reviewer did not investigate whether the conditions are equivalent). For if Pedersen's condition is not satisfied

then there are segments T_1, T_2 and values α', t, β' as above such that $\varphi(t) > \max(\varphi(\alpha'), \varphi(\beta'))$. If $x_1(\alpha')x_2(\alpha'') = \varphi(\alpha')$, $x_1(\beta')x_2(\beta'') = \varphi(\beta')$ and $x_1(t)x_2(t_0) = \varphi(t)$, let

$$(\alpha''-t'')/(\alpha''-\beta'') = (\alpha'-t)/(\alpha'-\beta').$$

Then $x_1(t)x_2(t'') \geqq x_1(t)x_2(t_0) > \max[x_1(\alpha')x_2(\alpha''), x_1(\beta')x_2(\beta'')]$ so that the author's condition also fails to hold. Consequently all results of the author are contained in Pedersen's paper, and even more so in G. *H. Busemann.*

Bukreev, B. Ya. Equidistant lines of constant geodesic curvature in the planimetry of Lobačevskiĭ. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 133–136. (Russian)

In the model of hyperbolic geometry with line element $ds^2 = R^2y^{-2}(dx^2+dy^2)$, $y>0$, consider a circle C: $x^2+(y-b)^2=h^2$, $b>h>0$. With the curves equidistant to C and their orthogonal trajectories as parameter lines, the line element becomes — this is the content of the paper —

$$ds^2 = du^2 + R^2\Big(h\cosh\frac{u}{R} \pm b\sinh\frac{u}{R}\Big)^2\Big(\frac{dv}{v^2+b^2-h^2}\Big)^2.$$

H. Busemann (Los Angeles, Calif.).

Maurin, K. Eingliedrige Gruppen der homogenen kanonischen Transformationen und Finslersche Räume. Ann. Polon. Math. **2** (1955), 97–102.

The author considers a canonical transformation: $\tilde{x}^i = \tilde{x}^i(x^j, p_j)$, $\tilde{p}_i = \tilde{p}_i(x^j, p^j)$ $(i, j = 1, \cdots, n)$ not as a transformation in a $2n$-dimensional phase space but as a transformation involving (1) the coordinates x^i of a point P of a manifold M_n of class C^1 and (2) the elements p_i of the tangent space $T(P)$ (of covariant vectors attached to P). It is shown that a 1-parameter group G_1 of certain homogeneous canonical transformations of this type induce a Finsler metric on M_n and conversely that a given Finsler metric on M_n determines a G_1, the geodesics being the trajectories of G_1. This result may be interpreted in terms of geometrical optics. The author indicates that this construction is applicable also to the general differentiable (but not necessarily finitely dimensional) manifolds of A. D. Michal [Bull. Amer. Math. Soc. **43** (1937), 394–401; **45** (1939), 529–563; MR **1**, 29]. *H. Rund.*

Kuiper, Nicolaas H., and Yano, Kentaro. On geometric objects and Lie groups of transformations. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 411–420.

A manifold X has a Λ-structure if it is endowed with a collection of coordinate systems which are related by transformations of Λ, where Λ is a pseudo-group of transformations of class C^s, $s>0$, in the corresponding number space R^n. Geometric object (g.o.) bundles B are defined with respect to Λ-structures in the base space, and Λ is (tacitly) assumed to be transitive in R^n. Not every fiber bundle can be a g.o. bundle because the g.o. bundles over cartesian products of lines and circles are topologically trivial. Every Λ-mapping in X is accompanied by an induced map (prolongation) in B. In particular, a Lie group H acting on X has an isomorphic prolongation in B. A cross section in B invariant under the prolongation of H (transitive on X) exists if and only if the isotropy subgroup I_x of H at $x \in X$, which acts on the fiber Y_x by prolongation, leaves invariant a point. It follows that on the group manifold of a Lie group all g.o. bundles are topologically trivial, and admit a left-invariant cross section through every point of B. Lie derivatives and Lie differentials are defined in the usual fashion, and the well-known theorem is proved that a necessary and sufficient condition for a g.o. field to be invariant under the prolongation of a connected Lie group acting on X is the vanishing of the Lie differential.

A. Nijenhuis (Chicago, Ill.).

Horie, Nobuo. On cyclic points of group-spaces. Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math. **29** (1955), 35–41.

Some variations, for transformation groups, on the fact that if g is an element of a connected Lie group then the inner automorphism by g is the identity if and only if $dL_g = dR_g$, L_g and R_g being left and right translation by g. *W. Ambrose* (Cambridge, Mass.).

See also: Jörgens, p. 493.

Riemannian Geometry, Connections

Kurita, Minoru. On the holonomy group of the conformally flat Riemannian manifold. Nagoya Math. J. **9** (1955), 161–171.

The vanishing of the conformal curvature tensor implies that the curvature 2-forms take the form $\Omega_{ij} = (\lambda_i+\lambda_j)\omega_i \wedge \omega_j$, where the numbers $(n-2)\lambda_i$ are the eigenvalues of $(R_{ij}-\delta_{ij}R/2(n-1))$. A discussion of this formula, aimed at finding a group $SO(k)$ contained in the local homogeneous holonomy group H_p at p for maximal k gives the following results: If $H_p \subset SO(n)$, $H_p \neq SO(n)$, for all p, then $H_p = SO(k)\times SO(n-k)$, or $= SO(n-1)$, or $=\{e\}$ for all points of the (connected, n-dimensional) manifold M; then also the (global) homogeneous restricted holonomy group is one of these groups, and M is locally a product space. If in some points $p \in M$, $H_p = SO(n)$, then the set of all p with this property is closed. The existence of an example of a space in which $H_p = SO(n)$ for some p, and $H_p = SO(k)\times SO(n-k)$ for other p is asserted.

A. Nijenhuis (Chicago, Ill.).

Komar, Arthur. Degenerate scalar invariants and the groups of motion of a Riemann space. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 758–762.

The paper begins with the following statement: If an n-dimensional Riemannian manifold admits n functionally independent scalars — constructed from the metric — it clearly cannot admit a continuous group of motions. The statement is false, since any such manifold can admit n functionally independent scalars. The text indicates that the author has confused the terms "scalars" and "scalar invariants of a group of transformations." He then gives an unconvincing proof of the apparently meaningless theorem: If a space admits at most $n-r$ functionally independent scalars, it will admit groups of motions in r independent directions.

C. B. Allendoerfer (Seattle, Wash.).

Teleman, C. Sur certains espaces symétriques. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 731–734. (Romanian. Russian and French summaries)

A V_{4p}^{*} is the space defined by identifying on the sphere S_{4p+3} in a euclidean E_{4p+4} all the points obtained from each other by the transformations of a certain motion group of three parameters G_3. The author shows that the V_{4p}^{*} can be considered as a non-holonomous manifold V_{4p+3}^{4p} defined by three Pfaff equations on S_{4p+3}. The

$V_{4p}{}^{4}$ is symmetric and its metric is given in its real form as well as in terms of p quaternionic variables.

R. Blum (Saskatoon, Sask.).

Vrănceanu, G. Sur les invariants des espaces A_2 à connexion linéaire. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6** (1954), 779–787. (Romanian. Russian and French summaries)

Given a space A_2 with connection $\Gamma_{jk}{}^{i}=A_{jk1}{}^{i}x^{1}+C_{jk}{}^{i}$, where $A_{jk1}{}^{i}$ and $C_{jk}{}^{i}$ are constants, then the $A_{jk1}{}^{i}$ constitute a tensor with respect to linear transformations of x^1, x^2. A relative symmetrical tensor V_{ijk1} is defined by

$$V_{1111}=A_{111}{}^{2},\quad V_{2222}=-A_{222}{}^{1},\quad V_{1112}=\tfrac{1}{3}(A_{(112)}{}^{2}-A_{(112)}{}^{1}),$$

$$V_{1112}=\tfrac{1}{3}(A_{(112)}{}^{2}-A_{111}{}^{1}),\ V_{1222}=\tfrac{1}{3}(A_{222}{}^{2}-A_{(122)}{}^{1}).$$

This tensor V can be used to classify the A_2 depending on the reality of the four lines determined by the equation $V_{ijk1}x^{i}x^{j}x^{k}x^{1}=0$.

D. J. Struik (Cambridge, Mass.).

Haimovici, M. Sur les espaces des familles de transformations de variable simplement transitive sans courbure de l'espèce III. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **1** (1950), 53–66. (Romanian. French summary)

Verf. zeigte in einer Reihe von Arbeiten [Ann. Sci. Univ. Jassy. Sect. I. **30**, 1–36 (1948); Disquisit. Math. Phys. **6** (1948), 81–128; MR **10**, 481], wie man einer Schar von einfach transitiven Transformationen eine Übertragungsgeometrie zuordnen kann. In der vorliegenden Arbeit wendet er diese Resultate auf denjenigen Spezialfall an, in dem die Krümmungstensoren dieser Geometrie verschwinden. In diesem Falle können die Transformationsgleichungen explizit angegeben werden.

O. Varga (Zbl **45**, 437).

Teleman, C. Sur les groupes de rotations. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6** (1954), 771–777. (Romanian. Russian and French summaries)

Dans cette note on résume la démonstration du théorème suivant, donné dans un autre travail [Acad. R. P. Romîne. Stud. Cerc. Mat. **5** (1954), 143–171; MR **16**, 624]: Le groupe de rotations G_p en $2p$ variables, ayant une transformation distinguée, est le sous-groupe non séparable maximum du groupe complet des rotations en $2p$ variables. On en déduit que les espaces de Vrănceanu $V_{2p}(\lambda)$ sont des espaces irréductibles à courbure variable ayant un groupe maximum de mouvements. Un espace homogène V_{2p+1} à courbure variable peut avoir un groupe ayant au plus $(p+1)^2$ paramètres, le maximum étant atteint par certains espaces dont la métrique est la somme de la métrique d'un espace $V_{2p}(\lambda)$ et du carré d'une forme de Pfaff

$$\omega=x^{1}dx^{2}-x^{2}dx^{1}+\cdots+x^{2p-1}dx^{2p}-x^{2p}dx^{2p-1}.$$

(Résumé de l'auteur.)

D. J. Struik.

Sun, J. Tseying. Geometrical approach to the equivalence problem of generalized spaces. J. Chinese Math. Soc. (N.S.) **1** (1951), 333–342. (Chinese summary)

This paper deals with the equivalence of pairs of metric (affine, or projective) spaces. Such a space is called "n-flat" if it is flat in the usual sense. It is "$(n-1)$-flat" if it contains a V_{n-1} which is itself flat, etc. A space "deviates" from a Euclidean space with "order one" if it is n-flat, with "order two" if it is both n-flat and $(n-1)$-flat, etc. The author claims to have proved the theorem: Two spaces are equivalent if they are of the same order. He then claims to have proved, that, for example, two metric spaces are equivalent if the equations

$$\bar{g}_{ij}=g_{pq}\frac{\partial x^{p}}{\partial\bar{x}^{i}}\frac{\partial x^{q}}{\partial\bar{x}^{j}};\quad \bar{B}_{ijk1}=B_{pqrt}\frac{\partial x^{p}}{\partial\bar{x}^{i}}\cdots\frac{\partial x^{t}}{\partial\bar{x}^{1}}$$

are consistent for determination of the variables $\bar{x}^i$ as functions of x^i. The reviewer regrets that this remarkable simplification of the classical theorem does not seem to be firmly established.

C. B. Allendoerfer.

Sun, J. Tseying. A remark on the equivalence problem for affine and metric spaces. J. Chinese Math. Soc. **2** (1953), 133–138. (Chinese. English summary)

An analytic proof for the equivalence of two affine spaces and two metric spaces in terms of their curvature tensors was given by the author in the paper reviewed above by making use of a known theorem concerning differential equations of mixed system [L. P. Eisenhart, Non-Riemannian geometry, Amer. Math. Soc. Colloq. Publ., v. 8, New York, 1927, p. 14]. In the present note we give a proof without any application of the theorem just mentioned, instead of which some identities in curvature tensors and its covariant derivations [Tôhoku Math. J. **43** (1937), 261–266; **44** (1937), 32–38] are used.

Autoreferat (Zbl **51**, 128).

See also: Bott, p. 521; Finzi, p. 544; Horváth und Moór, p. 544; Takeno, p. 545; Pihl, p. 546; Fujinaka, p. 546; Kondo, p. 546.

Complex Manifolds

★ **Bremermann, H. J. Holomorphic continuation of the kernel function and the Bergman metric in several complex variables.** Lectures on functions of a complex variable, pp. 349–383. The University of Michigan Press, Ann Arbor, 1955. $10.00.

On désigne par $K_G(z,\bar{t})=K_G(z_1,z_2;\bar{t}_1,\bar{t}_2)$ le noyau de Bergman d'un domaine G univalent de C^2; K_G est holomorphe des 4 variables complexes $z=(z_1,z_2)$; $\bar{t}=(\bar{t}_1,\bar{t}_2)$ dans $G_z\times G_{\bar{t}}$; il peut être prolongé dans l'enveloppe d'holomorphie $\mathfrak{H}(G_z\times G_{\bar{t}})$ qu'est égale à $\mathfrak{H}(G_z)\times\mathfrak{H}(G_{\bar{t}})$. Pour un polyèdre analytique P borné, $K_P(z,\bar{z})=K_P(z_1,z_2;\bar{z}_1,\bar{z}_2)\to+\infty$ quand z tend vers la frontière F de G. Un domaine d'holomorphie quelconque peut alors être approché de l'intérieur par des domaines $K_{P_\nu}(z,\bar{z})<M$, à frontière analytique réelle. Cependant il existe des domaines d'holomorphie G pour lesquels $K_G(z,\bar{z})$ ne tend pas vers $+\infty$ quand z tend vers F, tel le cas $(n=1)$ $G=[0<|z|<1]$, K_G coincidant alors avec le noyau relatif à $[|z|<1]$. Une condition suffisante pour que $K_G(z,\bar{z})$ tende vers $+\infty$ en dehors de tout compact de G est qu'il existe $f^M(z_1,z_2)\in L^2(G)$, avec $|f|\to+\infty$ quand $z\to z^M\in F$ et ceci pour tout point $M\in F$, G étant de plus borné. Il en est ainsi si G est strictement pseudo-convexe en tout point de F, celle-ci étant régulière. Si $K_G(z,\bar{t})$ est holomorphe dans $B_z\times B_{\bar{t}}$ avec $B\supset G$, toute fonction holomorphe $f\in L^2(G)$ se prolonge dans B. Dans le cas d'un domaine d'holomorphie G qui est limite de domaines d'holomorphie contenant $G+F$ (donc dans le cas où G n'a pas de „Nebenhülle"), il est toutefois impossible de prolonger $K_G(z,\bar{z})$ au delà de G comme fonction analytique réelle. Le mémoire se termine par la remarque que dans le cas où G est borné, différent de $\mathfrak{H}(G)$, tout point de F intérieur à $\mathfrak{H}(G)$ est à distance finie, pour la métrique de Bergman de G, celle-ci se prolongeant à $\mathfrak{H}(G)$.

P. Lelong (Paris).

Stoll, Wilhelm. Über meromorphe Modifikationen. III. Streueigenschaften analytischer und meromorpher Modifikationen. Math. Z. **62** (1955), 189–210.

On poursuit l'étude d'une modification méromorphe $\mathfrak{M}: G \to H$, G et H étant des variétés à structure analytique complexe, de dimension complexe 2 [voir Math. Z. **61** (1954), 206–234; **61** (1955), 467–488; MR **16**, 689, 813] étude faite dans le but d'étendre le théorème de H. Hopf relatif aux modifications holomorphes [voir Comment. Math. Helv. **29** (1955), 132–156; MR **16**, 813]. On note toujours

$$\mathfrak{M} = \mathfrak{M}[G, A, M, \tau; H, B, N, v]$$

la modification avec $\tau: A = G - M \to B = H - N$; $v = \tau^{-1}$. On établit ici que si $\mathfrak{M}$ et $\mathfrak{M}^{-1}$ sont ouvertes et méromorphes, τ étant singulière en $P_0 \in M$, l'ensemble des éclatements $\Sigma(P_0)$ contient un ensemble analytique $C \subset H$, de dimension complexe 2 en tous ses points. L'ensemble D des points de C où v est singulière n'a pas de point d'accumulation dans H. Pour $Q \in C - D$, on a $v(Q) = P_0$, et v est analytique dans l'ouvert $B \cup (C - D)$. Si de plus G et H sont à base dénombrable de voisinages, l'ensemble des points singuliers de τ n'a pas de point d'accumulation sur G. Le résultat essentiel obtenu est qu'à une singularité $P_0 \in N$ de τ correspond biunivoquement un ensemble analytique $C(P_0) \subset \Sigma(P_0) \subset N$, de dimension complexe 2 en tous ses points et tel que v soit holomorphe sur $C(P_0)$ sauf en des points isolés. *P. Lelong* (Paris).

Stoll, Wilhelm. Über meromorphe Modifikationen. IV. Die Erzeugung analytischer und meromorpher Modifikationen zwischen kompakten Mannigfaltigkeiten durch σ-Prozesse. Math. Ann. **130** (1955), 147–182.

Le but du mémoire est l'extension du théorème de H. Hopf [Comment. Math. Helv. **29** (1955), 132–156; MR **16**, 813] à des modifications méromorphes $G \to H$ entre variétés compactes et, sous certaines conditions, entre variétés ouvertes. Les notations sont celles des mémoires précédents [Math. Z. **61** (1954), 206–234; 467–488 (1955)]. $\mathfrak{M}[G, A, M, \tau; H, B, N, v]$ est la modification $G \to H$; τ est la représentation pseudo-conforme de $A = G - M$ sur $B = H - N$; $v = \tau^{-1}$; M et N sont les ensembles modifies sur G et H respectivement. On suppose G et H de dimension complexe 2, les ensembles M et N étant contenus dans des ensembles analytiques M^*, N^*, de dimension complexe au plus 1. On précise d'abord quelques propriétés caractéristiques des modifications σ_P qui consistent à remplacer un point P par l'espace projectif complexe des directions complexes issues de P; opérer des modifications σ en tous les points d'un ensemble M satisfaisant à l'hypothèse précédente constitue bien une modification analytique, M^* étant supposé contenu dans un voisinage ouvert de M sur G.

On établit la généralisation suivante du théorème de H. Hopf, celui-ci étant relatif au cas où M est un point: si N est un ensemble ne contenant qu'un nombre fini de variétés irréductibles de dimension complexe 1, il existe une suite finie $H_0 = G, H_1, \cdots, H_s = H$ de variétés permettant de passer de G à H; le passage de H_q, H_{q+1} se fait en exerçant des modifications σ_P (éclatement de Hopf) aux points d'un ensemble fini M_q et il existe une représentation pseudo-conforme de H sur H_q dont le degré est fini et va en décroissant. Pour qu'une modification $\mathfrak{M}$ soit une modification σ_p, il faut et il suffit que $\mathfrak{M}$ soit ouverte, M réduit à un point P unique, et que N soit une variété irréductible de dimension complexe 1 dans H.

On étudie ensuite des modifications méromorphes, avec l'hypothèse que M et N ne contiennent au plus qu'un nombre fini de variétés irréductibles de dimension complexe 1. On établit que, si $\mathfrak{M}$ et $\mathfrak{M}^{-1}$ sont ouvertes, il existe deux suites finies $\mathfrak{M}_q$, $\overline{\mathfrak{M}}_q$ de modifications consistant en des éclatements $P \to \sigma_P$ opérés sur des ensembles finis M_q, $\overline{M}_q$ la suite $\mathfrak{M}_q$ faite à partir de G et la suite $\overline{\mathfrak{M}}_q$ faite à partir de H conduisant à deux variétés identiques, à une correspondance pseudo-conforme près. L'énoncé s'applique aux variétés A et H compactes de dimension complexe 2. *P. Lelong* (Paris).

Siegel, Carl Ludwig. Meromorphe Funktionen auf kompakten analytischen Mannigfaltigkeiten. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1955**, 71–77.

Soient $\mathfrak{T}$ une variété analytique-complexe, compacte, de dimension complexe n, et K le corps des fonctions méromorphes sur $\mathfrak{T}$. Ce court article donne, en trois pages, une démonstration complète, et élémentaire, des deux théorèmes suivants (connus). Théorème 1: $n+1$ éléments quelconques de K sont algébriquement dépendants. Théorème 2: si $f_1, \cdots, f_n$ sont des éléments algébriquement indépendants de K, K est extension algébrique de degré fini du corps des fonctions rationnelles de $f_1, \cdots, f_n$.

La seule connaissance préalable exigée du lecteur est celle-ci: l'anneau des fonctions holomorphes en un point est factoriel. La démonstration de Siegel se rattache à une démonstration de Serre [Séminaire H. Cartan, 1953–54, exposée II], elle-même inspirée d'un théorème de Bochner [J. Indian Math. Soc. (N.S.) **16** (1952), 1–6; MR 13, 932]. Mais Serre utilisait le théorème de Cousin (i.e.: tout diviseur, dans un polydisque, peut être défini par une fonction méromorphe); Siegel évite le théorème de Cousin pour prouver le th. 1, puis, grâce au th. 1, démontre un cas particulier du th. de Cousin dont il a besoin pour le th. 2.

Signalons que le résultat de Serre vaut pour un cas plus général: celui où $\mathfrak{T}$ analytique-complexe n'est pas compacte, et où K est le corps des fonctions méromorphes invariantes par un groupe G d'automorphismes de $\mathfrak{T}$ tel qu'il existe un compact de $\mathfrak{T}$ rencontrant toute classe modulo G; dans cette situation, $\mathfrak{T}/G$ n'est pas forcément une variété analytique-complexe compacte.

L'auteur consacre deux pages à un intéressant exposé historique de l'ensemble des recherches dont les théorèmes 1 et 2 marquent un aboutissement, à commencer par les travaux de Riemann sur les périodes des intégrales abéliennes. *H. Cartan* (Paris).

Umezawa, Toshio. On some systems of regular functions. J. Math. Soc. Japan **7** (1955), 137–165.

On considère à la suite de E. Peschl et F. Erwe [Math. Ann. **126** (1953), 185–220; MR **15**, 520] les systèmes $f_i(z)$ de n fonctions analytiques de m variables complexes $z = (z_1, \cdots, z_m)$, fonctions qu'on interprète comme les coordonnées d'un vecteur $F(z)$. Diverse formules sont données, notamment une formule d'interpolation dans un polygone convexe. Pour $m = 1$, $n \geq 2$, on étudie les régions où la représentation $F(z)$ dans C^n est p-valente ou au plus p-valente; on étend ainsi à $F(z) = \sum_0^\infty A_k z^k$ ($A_k = a_k^i$, $1 \leq i \leq n$) des énoncés connus dans le cas d'une fonctions $f(z)$ p-valente d'une variable. *P. Lelong.*

Fuks, B. A. The geometric meaning of the argument of the Jacobian of a pseudoconformal mapping in the space of n complex variables. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1(63), 179–182. (Russian)

Let (*) $w_k = w_k(z_1, \cdots, z_n)$ ($k = 1, \cdots, n$) be regular at

the point $P(z_1^0, \cdots, z_n^0)$, and let the Jacobian $J=J(z_1, \cdots, z_n)$ be different from 0 at P. The author studies the linear mapping $w_k=\sum_{s=1}^{n} a_{ks}z_s$ $(k=1, \cdots, n)$ associated with (*), where $a_{ks}=(\partial w_k/\partial z_s)_P$, and shows that, if the roots λ_s $(s=1, \cdots, n)$ of the equation $\det(a_{kr}-\lambda\bar{a}_{rk})=0$ are all of modulus 1, $\lambda_s=e^{i\vartheta_s}$, then $\vartheta_1+\cdots+\vartheta_n=\arg J \pmod{\pi}$.

A. J. Lohwater (Helsinki).

Wang, Hsien-Chung. Complex parallisable manifolds. Proc. Amer. Math. Soc. **5** (1954), 771–776.

A complex analytic manifold M of complex dimension n is said to be complex parallisable if it admits n holomorphic l-forms ω^l linearly independent at every point. Then we have $d\omega^l=\sum C_{jk}{}^l\omega^j\wedge\omega^k$, with holomorphic $C_{jk}{}^l$. From now on, M is supposed to be compact. Then the $C_{jk}{}^l$ are constant and it follows easily that M is isomorphic to the quotient G/D of a simply connected complex Lie group G with constants of structure $C_{jk}{}^l$ by a discrete subgroup D. Moreover, the cohomology ring of the ring of holomorphic exterior differential forms on M is isomorphic to the cohomology ring of the complex Lie algebra of G. It follows in particular that M is kählerian if and only if it is a complex torus.

A. Borel (Zbl **56**, 154).

Lichnerowicz, André. Sur les groupes d'automorphismes de certaines variétés kähleriennes. C. R. Acad. Sci. Paris **239** (1954), 1344–1346.

En utilisant des méthodes développées antérieurement par lui [C.R. Acad. Sci. Paris **237** (1953), 695–697; Géométrie différentielle, Colloq. Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 171–184; MR **16**, 519], l'A. prouve les théorèmes suivants: Si V_{2n} est une variété kählerienne avec $\|R_{jk}\|\neq 0$, le plus grand groupe connexe d'isométries de V_{2n} préserve sa structure analytique. — Sur une variété complexe à métrique Einsteinniene $(R\neq 0)$ de dimension $2n$, l'équation $\Delta\varphi=\lambda\varphi$ n'admet pas de valeur propre $\lambda<R/n$ (φ scalaire); il existe un isomorphisme entre l'algèbre de Lie du groupe des automorphismes de la variété et une algèbre définie sur l'espace des solutions de l'équation $\Delta\varphi=(R/n)\varphi$. L'isomorphisme est donné par $Id\varphi=\xi$ (φ scalaire, ξ vecteur de Killing), l'algèbre sur les fonctions φ est donnée par $[\varphi, \psi]=F^{ij}\partial_i\varphi\partial_j\psi$, F_{ik} étant la forme alternée associée à la métrique.

H. Guggenheimer (Zbl **56**, 412).

Schouten, J. A., and Yano, K. On pseudo-Kählerian spaces admitting a continuous group of motions. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 565–570.

A theorem of Lichnerowicz [see the paper reviewed above] states that for a Kähler manifold V_{2n} every motion of the metric leaves invariant the complex structure if (1) V_{2n} is irreducible; (2) the Ricci tensor does not vanish; (3) n is even. The authors succeed in weakening the conditions to (0) V_{2n} is pseudo-Kählerian; (1) V_{2n} is irreducible; (2) the Ricci tensor does not vanish for n even; (3) n is odd or even. The proof involves considerations on Lie derivatives, on pseudo-analytic covariant vector fields and on linear subspaces on the null-cone of the complexification of the tangent space.

A. Nijenhuis (Chicago, Ill.).

See also: Teleman, p. 528; Morikawa, p. 533; Hirzebruch, p. 535.

Algebraic Geometry

Thalberg, Olaf M. Some properties of algebraic curves with a $(n-2)$-ple point. Avh. Norske Vid. Akad. Oslo. I. **1954**, no. 2, 8 pp. (1955).

Let p be an $(n-2)$-ple point, with ordinary branches, of an algebraic plane curve C_n of order n, let $t_1, t_2, \cdots, t_{2(n-1)}$ be the points of contact of the tangents from p to C_n, and let $s_1, s_2, \cdots, s_{n-2}$ be the further intersections of the tangents at p. Using the Jonquières involution with center at p and coincident curve C_n, the author studies the geometric relations between the points p, t_i, s_j.

G. B. Huff (Athens, Ga.).

Casteels, L. Extension d'un théorème de J. Bacharach sur les points d'intersection. Meded. Kon. Vlaamse Acad. Kl. Wetensch. **16** (1954), no. 10, 18 pp. (Dutch. French summary)

Let B_p and C_q be fixed algebraic curves of resp. degrees p and q, whose intersection consists of a finite number of points. Let $G(pq-\varepsilon)$ be any group of $pq-\varepsilon$ distinct common points of B_p and C_q. Assume $p\leq q$ and let n be an integer satisfying $q\leq n\leq p+q-\varepsilon+2$; it is shown: (i) The fact that an A_n contains $G(pq-\varepsilon)$ is expressed by $pq-\frac{1}{2}(p+q-n-1)(p+q-n-2)$ independent conditions for the coefficients of A_n. (ii) Every A_n which contains $G(pq-\varepsilon)$ passes through all common points of B_p and C_q, and $A_n=bD_{n-p}B_p+cE_{n-q}C_q$ with constants b and c. The particular case when the intersection of B_p and C_q consists of pq distinct points was found by Bacharach [Math. Ann. **26** (1885), 275–299, pp. 286–288].

F. J. Terpstra (Pretoria).

Bonera, Piero. Sui nodi delle curve gobbe razionali dotate di quaterna armonica di punti d'iperosculazione. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 81–86.

A rational skew curve C_n of order n in complex S_4 with four points of hyperosculation forming a harmonic set has a simple parametric representation. The author uses this to show that: if n is odd, such a curve has no nodes; if n is even, such a curve has at least $n-3$ or $n-2$ nodes, according as n is or is not divisible by 4; and that if n is even and not divisible by 3, there can be no more nodes than the minimum number. There are further results depending on the prime factors of n. *G. B. Huff.*

Godeaux, Lucien. Sur les involutions cycliques appartenant à la surface des couples de points d'une courbe algébrique. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 1094–1100.

Godeaux, Lucien. Surfaces algébriques tracées sur un cône de Veronese. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 863–869.

Morgantini, Edmondo. Sulle varietà tridimensionali contenenti una rete lineare di superficie razionali irriducibili. Rend. Sem. Mat. Univ. Padova **23** (1954), 100–126.

Im Anschluß an Arbeiten von G. Fano [Ann. Mat. Pura Appl. (3) **24** (1915), 49–88], U. Morin [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **29** (1939), 116–118; Ann. Mat. Pura Appl. (4) **18** (1939), 147–171; MR **1**, 83], und M. Baldassarri [Rend. Sem. Mat. Univ. Padova **20** (1951), 135–152; Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12** (1952), 390–397; MR

13, 976; 14, 790]verbessert Verf. eine bereits von F. Enriques [Math. Ann. 49 (1897), 1–23] aufgestellte Klassifikation dahingehend, daß eine dreidimensionale algebraische Mannigfaltigkeit V_3, welche ein lineares Netz $|\Phi|$ von rationalen Flächen trägt, immer birational auf einen der folgenden Typen zurückgeführt werden kann: (I) auf den einfachen S_3; (II) auf den doppelten S_3 mit einer Verzweigungsfläche: (A) der Ordnung $2m$, die einen $(2m-2)$-fachen Punkt besitzt, oder (B) der Ordnung 6 mit zwei benachbarten 3-fachen Punkten. Der Beweis geht aus von einem projektiven Modell W_3 der V_3 im S_4, das so konstruiert ist, daß die charakteristischen Kurven des Netzes $|\Phi|$ irreduzibel sind, und benützt die Eigenschaften der sukzessiven adjungierten Systeme zu den hyperebenen Schnitten der einzelnen Φ. Durch eingehende Diskussion der hier möglichen Fälle wird das Resultat abgeleitet. *W. Gröbner* (Zbl 55, 390).

Marchionna Tibiletti, Cesarina. Trecce relative a forme canoniche del gruppo di monodromia. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 25–40.

La construction des tresses algébriques représentatives d'une courbe algébrique $f(x, y)=0$ est déjà connue lorsque le groupe de monodromie de la fonction $y(x)$ définie par la courbe est ramené à l'une des formes canoniques:

a) $(12)^2(13)^2$, $(1n)^2(23)^2$, $(2n)^2$, $(n-1n)^2$;

b) $[(12)(13)\cdots(1n)]^n$.

L'A. établit à partir des règles classiques de l'algèbre des tresses [M. Dedò, mêmes Rend. (3) 14(83) (1950), 227–258; MR 13, 973], les tresses correspondantes aux formes canoniques suivantes, cas particuliers de la forme de Lüroth-Clebsch:

c) $(12)^2(23)^4$, $(n-2\,n-1)^{2n-4}(n-1\,n)^{2n-2}$;

d) $(12)^2(23)^3$, $(n-2\,n-1)^3(n-1\,n)^{2p+2}$.

Cette dernière, dont la tresse se déduit de la précédente, correspond à une forme limite donnée par une courbe hyperelliptique. L'A. montre que la tresse relative à une courbe f passant r fois par le point impropre de l'axe Oy s'obtient à partir de la tresse générale relative à la forme d) par suppression des r premiers et des $r(r+1)/2$ premiers traits. *B. d'Orgeval* (Dijon).

Derwidué, L. Sur les variétés exceptionnelles. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **26** (1951), no. 5, 40 pp.

L'A. étend aux variétés algébriques privées de singularités les notions de courbes exceptionnelles de première espèce (variétés exceptionnelles propres) et de seconde espèce (variétés exceptionnelles impropres). Il montre ensuite qu'il est toujours possible de transformer une variété privée de singularités en une variété privée de variétés exceptionnelles propres, mais sur laquelle il peut exister des variétés exceptionnelles impropres, au moyen d'un nombre fini de transformations élémentaires. L'A. donne ce nom à des transformations birationnelles obtenues de la manière suivante: Sur une variété algébrique V_n, on considère les variétés W_{n-1} passant par une sous-variété et formant un système simple. En rapportant projectivement les variétés W_{n-1} aux hyperplans d'un espace de dimension convenable, V_n se transforme birationnellement en une variété V_n' par une transformation élémentaire. *L. Godeaux* (Zbl 45, 420).

Derwidué, L. A propos des variétés exceptionnelles. Acad. Roy. Belg. Bull. Cl. Sci. (5) **39** (1953), 70–76.

Etude des variétés exceptionnelles qui peuvent exister dans une transformation birationnelle entre deux variétés algébriques privées de singularités.

L. Godeaux (Zbl 51, 121).

★ **Burniat, Pol. Sur les surfaces canoniques de genres $p_g=4$, $p^{(1)}\geqq 11$.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 271–281. Casa Editrice Perrella, Roma, 1953.

Dans le but de déterminer des surfaces canoniques, c'est-à-dire des surfaces de S_3 dont le système canonique est formé par les sections planes. l'A. construit des surfaces doubles répondant à la question et montre que, sous certaines conditions, ces surfaces appartiennent à des systèmes continus de surfaces en général irréductibles, qui sont également canoniques, et dont le genre linéaire a l'une des valeurs 11, 13, 15, 17. On n'était parvenu avant qu'à des surfaces canoniques de genre linéaire inférieur à 11. *L. Godeaux* (Zbl 50, 373).

Benedicty, Mario. Sopra alcuni appunti inediti di Fabio Conforto. Rend. Mat. e Appl. (5) **14** (1955), 487–509.

Après la mort du prof. F. Conforto de l'Université de Rome, on a trouvé dans les manuscrits qu'il a laissé une foule de questions énoncées et de travaux commencés et pas encore finis. Le but de l'exposition de M. Benedicty, à la quelle on doit rapprocher une exposition analogue de M. Rosati [résumée ci-dessous], est de faire connaître d'une manière ordonnée ce qu'il y a d'important et d'utile dans les papiers de F. Conforto. Tandis que M. Rosati s'est intéressé des questions se rapportant plus directement à la théorie des fonctions abéliennes, l'A. s'occupe ici surtout de tout ce qui se rapporte aux fonctions quasi-abéliennes. Il s'agit de questions qui peuvent être classées en trois directions principales, c'est-à-dire: théorie arithmétique des fonctions quasi-abéliennes; fonctions quasi-abéliennes du point de vue fonctionnel, théorie des fonctions quasi-abéliennes modulaires. On doit ajouter trois questions de géometrie algébrique: sur les correspondances plurilinéaires, sur les transformations birationnelles entre deux plans, sur les faisceaux de surfaces d'ordre $2r$ possédant 8 points de base n-uples sur une courbe dégénérée du 4ème ordre et de la 1re espèce. A la fin l'A. donne une solution de deux des questions arithmétiques rappelées plus haut sur les fonctions quasi-abéliennes. *E. G. Togliatti* (Gênes).

Rosati, Mario. Funzioni abeliane ed abeliane modulari nelle lezioni e nei manoscritti inediti di Fabio Conforto. Rend. Mat. e Appl. (5) **14** (1955), 696–711.

Cette exposition doit être rapprochée à une autre de M. Benedicty, le contenu de laquelle est résumé ci-dessus; il s'agit des manuscrits laissés par F. Conforto, et des questions que ce savant y avait énoncées ou partiellement résolues. Tandis que M. Benedicty s'est occupé surtout des questions sur les fonctions quasi-Abéliennes, M. Rosati s'interesse ici des questions sur les fonctions modulaires Abéliennes et les fonctions Abéliennes. Le problème central sur les fonctions modulaires Abéliennes est la construction d'un ensemble I dont les points représentent les corps différents de fonctions Abéliennes dépendant d'un nombre donné p de variables; cela revient à classifier les matrices de Riemann $\omega^{(p,2p)}$, de genre p, par rapport à la relation d'équivalence $\omega'=\alpha\omega A$, où $\alpha^{(p,p)}$ n'est pas singulière et $A^{(2p,2p)}$ est une matrice entière unimodulaire. S

les diviseurs élémentaires de ω sont donnés, on est conduit à l'étude d'un groupe modulaire proprement discontinu et de son domaine fondamental R^*; les questions posées ou abordées par F. Conforto se rapportent surtout à l'étude de R^* et de sa frontière. Il y a plusieurs manuscrits sur la théorie des fonctions Abéliennes (en vue surtout des applications aux variétés de Picard à p dimensions); en particulier sur la classification des matrices de Riemann impures. *E. G. Togliatti* (Gênes).

Rosati, Mario. Sulle varietà di equazione $\omega M\omega_{-1}=0$ con M matrice modulare. Rend. Mat. e Appl. (5) **14** (1955), 712–728.

W étant une variété algébrique avec l'irrégularité superficielle $p>0$, soient V_p, V_p' les deux variétés de Picard à p dimensions liées à W selon F. Severi et selon G. Castelnuovo. Les deux matrices de Riemann ω, ω' correspondant à V_p, V_p' ne sont pas toujours équivalentes, de sorts que V_p, V_p' ne sont pas toujours birationnellement identiques. En écrivant ω, ω' sous la forme normale de F. Conforto:

$$\omega=\|\Delta^{-1}\Omega\|,\quad \omega'=\|\Delta^{-1}\Omega'\|$$

(Δ, Δ', que l'on appelle les niveaux de ω, ω' sont deux matrices diagonales avec les diviseurs élémentaires δ_i, δ_i', et Ω, Ω' sont deux matrices complexes symétriques dont les parties imaginaires sont formées avec les coefficients de deux formes quadratiques définies positives), on a l'équivalence de ω, ω' si leurs niveaux sont égaux à 1 (c'est-à-dire si $\delta_i=\delta_i'=1$); si cette condition n'est pas vérifié, alors ω, ω' sont équivalentes seulement si ω est une solution ω^* de l'équation $\omega M\omega_{-1}=0$, $M^{(2p,2p)}$ étant une matrice entière telle que $|M|=1$. L'A. considère ici les deux cas particuliers suivants, où M est émisymétrique le cas $M=\begin{vmatrix}0 & u\\ -u & 0\end{vmatrix}$, et le cas $M=\begin{vmatrix}0 & J\\ -J & 0\end{vmatrix}$; ici 0, u sont la matrice nulle et la matrice unité de l'ordre p, J est une matrice de l'ordre p dont tous les éléments sont nuls à l'exclusion de ceux de la diagonale sécondaire qui sont tous égaux à 1. Ces exemples montrent d'un côté l'existence de matrices ω^* pour un niveaux Δ quelconque arbitrairement donné; et, d'un autre côté, qu'il existent pour tout niveau $\Delta\neq u$ des solutions ω^* n'appartenant pas au niveau unitaire. On trouve aussi que la dimension de la variété des solutions ω^* dépend de M, et, une fois que M est donnée, elle dépend encore des propriétés arithmétique des diviseurs de Δ. *E. G. Togliatti* (Gênes).

Morikawa, Hisasi. Cycles and multiple integrals on abelian varieties. Proc. Japan Acad. **31** (1955), 317–320.

Certain computations with period matrices produce "some relations between cycles and multiple integrals which are similar to the known relations between divisors and simple integrals on abelian varieties".

M. Rosenlicht (Evanston, Ill.).

Morikawa, Hisasi. Cycles on algebraic varieties. Nagoya Math. J. **9** (1955), 173–180.

The theory of harmonic integrals is used to derive a number of results on the period matrices for various dimensions of an algebraic variety V^n. A new birational invariant is derived, this being the number of linearly independent rational homology classes of dimension $(2n-2)$ which contain algebraic cycles modulo the subgroup consisting of those having zero intersection number with pairs of $(2n-1)$-cycles and a generic linear section of complex dimension two. *M. Rosenlicht.*

Hodge, W. V. D., and Atiyah, M. F. Integrals of the second kind on an algebraic variety. Ann. of Math. (2) **62** (1955), 56–91.

This paper, whose main results were announced earlier [C.R. Acad. Sci. Paris **239** (1954), 1333–1335; MR **16**, 951], is concerned with a study of integrals of the second kind on an algebraic variety, using as main tool the theory of stacks (or sheaves or faisceaux). Chapter I deals with simple and double integrals of the second kind on an algebraic variety of arbitrary dimension. The definition is expressed in terms of stacks, but, in the case of double integrals of the second kind, is shown to be equivalent to one of Picard-Lefschetz. The main theorem states that the number of independent 2-forms of the second kind is $R_2-\varrho$, where R_2 is the second Betti number and ϱ is the Picard number, the maximum number of inequivalent divisors. Emphasis is laid on geometric interpretations of stack-theoretic results and on relations with classical theory.

Chapter II begins with a summary of the Leray-Cartan spectral theory and is devoted to the general theory of meromorphic q-forms. Let V denote a non-singular irreducible algebraic variety of dimension m over the complex field, and let W be an algebraic subvariety of dimension $m-1$. Denote by $\Omega^q(*W)$ the stack of germs of q-forms holomorphic in $V-W$ and having polar singularities on W. Let $\Omega(*W)=\sum_q\Omega^q(*W)$, and let $\Omega(*)$ be the direct limit of $\Omega(*W)$, as W runs over the subvarieties of V. Similarly, denote by $\tilde{\Omega}^q(W)$ the stack of germs of complex-valued q-forms of class C^∞ on $V-W$, and having arbitrary singularities on W, and define $\tilde{\Omega}(W)$, $\tilde{\Omega}(*)$ in an analogous manner. The authors view as their main problem the study of the relationship between the analytic stack $\tilde{\Omega}(*W)$ and the stack $\tilde{\Omega}(W)$, the latter depending on the geometry of V relative to its subvarieties. By supposing W to be ample and taking a fundamental stack $F=\sum_p F^p$, where F^p is the stack of germs of C^∞ p-forms, the authors introduce two spectral sequences $E_r^{p,q}(W)$ and $\tilde{E}_r^{p,q}(W)$. The term $E_\infty^{p,q}(W)$ gives a filtration of the group $G^{p+q}(W)$, the residue class group of all meromorphic $(p+q)$-forms on V with no singularities except those on W, modulo the derived forms. Similarly, $\tilde{E}_\infty^{p,q}(W)$ gives a filtration of $H^{p+q}(W)\cong H^{p+q}(V-W,C)$, the $(p+q)$-th cohomology group of $V-W$ over the complex field. If W is moreover simple, $E_r^{pq}(W)$ and $\tilde{E}_r^{pq}(W)$ are isomorphic for $r\geqq 2$. From this one derives

$$\dim G_q^q(W)=\dim E_{q+1}^{q,0}(W)=\dim(w^*H^q(V,C)),\quad q\geqq 1,$$

where $w:V-W\to V$ is the injection map. The study of the general case will depend on the conjecture on the resolution of singularities of algebraic varieties by birational transformations. The authors derive several results based on this conjecture. These include an identification of the author's definition of forms of second kind and the definition in terms of residues. Finally, a filtration of $G^{pq}=\lim_W G^q(W)$ is suggested, which is birationally invariant and hence should lead to a birationally invariant spectral sequence. *S. Chern* (Chicago, Ill.).

Weil, André. On algebraic groups of transformations. Amer. J. Math. **77** (1955), 355–391.

Weil, André. On algebraic groups and homogeneous spaces. Amer. J. Math. **77** (1955), 493–512.

Eine Menge G, in der sowohl eine Gruppenstruktur als auch die Struktur einer algebraischen Mannigfaltigkeit erklärt ist [es handelt sich dabei um „abstrakte" Mannig-

faltigkeiten im Sinne des Verf.; vgl. A. Weil, Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ., v. 29, New York, 1946; MR 9, 303], heißt eine algebraische Gruppe, wenn die Gruppenoperationen $(x, y) \to xy$ und $x \to x^{-1}$ reguläre Abbildungen im Sinne der algebraischen Geometrie sind. In der ersten Arbeit charakterisiert Verf. bei vorgegebenem Konstantenkörper k diejenigen separablen Funktionenkörper K/k, welche eine (über k definierte) algebraische Gruppe als Modell besitzen. In seinem Buch über abelsche Mannigfaltigkeiten [Variétés abéliennes et courbes algébrique, Hermann, Paris, 1948; MR 10, 621] hatte er bereits Bedingungen angegeben, denen ein solcher Funktionenkörper notwendig genügen muß; bezüglich des Hinreichens dieser Bedingungen konnte er aber dort nur zeigen, daß eine geeignete Konstantenerweiterung von K/k eine algebraische Gruppe als Modell besitzt. Der hier erzielte Fortschritt beruht darin, daß die Notwendigkeit einer Konstantenerweiterung eliminiert wird und die erwähnten Bedingungen auch als hinreichend erwiesen werden.

Diese Bedingungen fordern, daß es einen Isomorphismus α von K in das (über k gebildete) zweifache unabhängige Körperkompositum $K \times K$ geben soll, und daß α zwei Eigenschaften besitzen soll, welche der Auflösbarkeit und der Assoziativität der Gruppenverknüpfung entsprechen (G1, S. 356; G2, S. 357). Das zugehörige Gruppenmodell G von K besitzt die Eigenschaft, daß für zwei über k unabhängige allgemeine Punkte $x, y \in G$ der natürliche Isomorphismus $K \times K \to k(x, y)$ den (zu K isomorphen) Körper $K\alpha$ auf den (ebenfalls zu K isomorphen) Körper $k(xy)$ abbildet; dadurch ist G bis auf Isomorphismen eindeutig bestimmt. Die vom Verf. angegebene Konstruktion von G bei vorgegebenen K, α ist im wesentlichen elementarer Natur; sie beginnt mit der Beobachtung, daß jedes beliebige Modell V von K, wenn es auch kein Gruppenmodell ist, jedenfalls eine offene Teilmenge (im Sinne der Zariskischen Topologie) enthält, welche sich wie eine offene Teilmenge eines zu K, α gehörigen Gruppenmodells verhält (Prop. 4, S. 365). Diese offenen Teilmengen, genannt „group chunks", werden zunächst für verschiedene V isoliert, und dann im Sinne der Theorie der abstrakten Mannigfaltigkeiten zusammengefügt, um das gesuchte Gruppenmodell G von K zu ergeben. Die Methode des Zusammenfügens erfordert zunächst eine Konstantenerweiterung; danach wird jedoch durch eine gesonderte Überlegung nachgewiesen, daß das so erhaltene Modell G bereits über k definiert ist. Bemerkenswert ist, daß diese Überlegung für einen endlichen Konstantenkörper k versagt, und daß für einen solchen der Beweis unter Zuhilfenahme der Riemannschen Vermutung geführt wird [siehe Lang und Weil, Amer. J. Math. 76 (1954), 819–827; MR 16, 398].

Gleichzeitig mit einer algebraischen Gruppe G betrachtet Verf. auch Darstellungen von G als Automorphismengruppe einer algebraischen Mannigfaltigkeit S; eine solche Darstellung soll algebraisch sein in dem Sinne, daß die Operatoranwendung $(x, s) \to xs$ eine reguläre Abbildung von $G \times S$ auf S ist. S heißt Darstellungsmannigfaltigkeit (transformation space) für G. Wird insbesondere S durch G transitiv permutiert, so heißt S homogen. Verf. charakterisiert bei vorgegebenem Konstantenkörper k und bei vorgegebener, über k definierter, algebraischer Gruppe G diejenigen separablen Funktionenkörper L/k, welche eine (über k definierte) Darstellungsmannigfaltigkeit für G als Modell besitzen. Diese Charakterisierung erfolgt wiederum durch geeignete Isomorphiebedingungen (TG1, TG2, S. 359). Auch für die Homogeneität von S wird eine notwendige und hinreichende Isomorphiebedingung angegeben (H, S.360); ist diese erfüllt, so ist S bis auf Isomorphie eindeutig bestimmt.

In der zweiten Arbeit verwendet Verf. diese Ergebnisse zunächst dazu, um nachzuweisen, daß die Faktorgruppe G/H einer algebraischen Gruppe G nach einem Normalteiler H, der sich aus endlich vielen Teilmannigfaltigkeiten von G zusammensetzt, in natürlicher Weise die Struktur einer algebraischen Gruppe besitzt. Ist H kein Normalteiler, sondern nur eine Untergruppe von G ist, so ist G/H wenigstens eine Darstellungsmannigfaltigkeit für G (Prop.2, S.495). Sind G und H beide über k definiert, so ist es auch G/H; hierin liegt der wesentliche Fortschritt gegenüber Nakano [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27 (1952), 55–66; MR 14, 201].

Der Rest der Arbeit ist den einfach transitiven homogenen Darstellungsmannigfaltigkeiten (e.t.h.Dm.) einer algebraischen Gruppe G gewidmet (principal homogeneous spaces). Es gelingt, bei vorgegebenem Konstantenkörper k und vorgegebenem G die separablen Funktionenkörper L/k, welche eine e.t.h.Dm. S für G als Modell besitzen, durch eine Isomorphiebedingung zu charakterisieren (TG 1′, S.498). Jedes solche L wird nach geeigneter Konstantenerweiterung zu dem Funktionenkörper K von G isomorph; besitzt S einen in k rationalen Punkt, so ist diese Konstantenerweiterung sogar unnötig. Zwei über k definierte e.t.h.Dm. S und S' für G mögen zur selben Klasse gerechnet werden, wenn es eine birationale, über k definierte, Abbildung φ von S auf S' gibt derart, daß für je zwei über k unabhängige allgemeine Punkte $x \in G$, $s \in S$ gilt: $\varphi(xs) = x\varphi(s)$. Falls G kommutativ ist, so kann die Menge der Klassen e.t.h.Dm. für G zu einer Gruppe gemacht werden, in welcher alle Elemente endliche Ordnung besitzen (Prop. 5, S.503ff.). Verf. diskutiert diese Ergebnisse dann noch für den Spezialfall, wo G die Mannigfaltigkeit der Divisoren nullten Grades einer singularitätenfreien algebraischen Kurve ist (Jacobische Mannigfaltigkeit), und er stellt den Zusammenhang mit den diesbezüglichen Resultaten von Chow her [Amer. J. Math. 76 (1954), 453–476; MR 15, 823; vgl. dazu auch Weil, Arch. Math. 5 (1954), 197–202; MR 15, 896].

P. Roquette (Princeton, N.J.).

Igusa, Jun-ichi. On some problems in abstract algebraic geometry. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 964–967.

Exemple, en caractéristique 2, d'une surface non singulière ayant les propriétés suivantes: la dimension g de sa variété de Picard est 1, son nombre $h^{1,0}$ de formes différentielles de première espèce est 2, et la déficience $h^{0,1}$ du système linéaire complet des sections de V par les hypersurfaces de degré élevé est 2. Ainsi la conjecture que la série caractéristique d'un système algébrique complet de diviseurs est complète est fausse en caractéristique 2 (plus généralement $p \neq 0$). D'autre part, lorsque V est définie sur un corps fini (ce qui est réalisable) ses polynomes de Poincaré définis au moyen de la fonction zèta [A. Weil, Bull. Amer. Math. Soc. 55 (1949), 497–508; MR 10, 592] et au moyen de la théorie des faisceaux [cf. J. P. Serre, Ann. of Math. (2) 61 (1955), 197–278; MR 16, 953] sont distincts.

La surface V est ainsi construite: soient A, A' deux courbes elliptiques (en caractéristique 2) et soit t un point d'ordre 2 de A; l'application $(a, a') \to (a+t, -a')$ de $A \times A'$ sur lui même est une involution birégulière d'ordre 2, et V est la variété des couples (non ordonnés) de points ho-

mologues pour cette involution. La variété de Picard de V est la courbe $A/(1, t)$, sa classe canonique est 0. On a $p_g(V)=1$, $p_a(V)=-1$.

Example analogue en caractéristique $p\neq 0,2$: on remplace A par une courbe elliptique, t par un point d'ordre p de A, et A' par une courbe de genre $\frac{1}{2}(p-1)$ admettant un automorphisme de dimension p. On peut aussi construire une variété non singulière d'ordre 3 telle que $g=0$ et que $h^{1,0}=3$; d'où dessurfaces non singulières telles que $g=0$ et que $h^{1,0}$ soit arbitrairement grand. *P. Samuel.*

Hirzebruch, Friedrich. Arithmetic genera and the theorem of Riemann-Roch for algebraic varieties. Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 110–114.

Ueber einer algebraischen Mannigfaltigkeit V_n, welche singularitätenfrei in einem komplexen projektiven Raum liegt, sei ein analytisches Faserbündel W mit dem komplexen Vektorraum C_q als Faser und der linearen Gruppe $GL(q, C)$ als Strukturgruppe gegeben. Die „Charakteristik" $\chi(V_n, W)$ ist definiert durch

$$\chi(V_n, W)=\sum_0^n (-1)^i \dim H^i,$$

wo H^i die $i$te Cohomologiegruppe von V_n mit Koeffizienten in der Garbe (faisceau) der lokalen analytischen Schnittflächen von W bedeutet; man weiss, dass $\dim H^i$ endlich ist. Es seien c_i bzw. d_i die Chernschen Klassen des Tangentialbündels von V_n bzw. des Faserbündels W und γ_i bzw. δ_i ihre formalen Wurzeln (d.h., mit einer Unbestimmten x, $\sum c_i x^i=\prod(1+\gamma_i x)$ und $\sum d_i x^i=\prod(1+\delta_i x)$). Der Hauptsatz der vorliegenden Note besagt, dass

$$\chi(V_n, W)=\varkappa\Big[e^{\delta_1}+e^{\delta_2}+\cdots+e^{\delta_q})\prod\frac{-\gamma_i}{e^{-\gamma_i}-1}\Big]$$

ist, wo $\varkappa$ den Wert der $2n$-dimensionalen Komponente von $[\cdots]$ auf dem natürlich orientierten Grundzyklus von V_n bedeutet. Spezialfälle: (1) Ist W das triviale C_1-Bündel über V_n, so ist $\chi(V_n, W)=\prod(V_n)$ das arithmetische Geschlecht von V_n, die rechte Seite das Toddsche Geschlecht $T(V_n)$ [vgl. F. Hirzebruch, dieselben Proc. **39** (1953), 951–956; MR **16**, 159], womit also die Gleichheit dieser beiden Geschlechter bewiesen ist. (2) Ist W ein beliebiges C_1-Bündel über V_n, f seine charakteristische Klasse, so gilt $\chi(V_n, W)=\varkappa\Big[e^{f+n/2}\prod\frac{\gamma_i/2}{\sinh \gamma_i/2}\Big]$. Da jeder Divisor D von V_n die charakteristische Klasse eines C_1-Bündels — ebenfalls D genannt — repräsentiert, enthält (2) Aussagen vom Typ des Riemann-Rochschen Satzes; sie umfassen alle seine bisher bekannten Verallgemeinerungen. Insbesondere erhält man in dem Falle, wo $D-K$ (K=kanonischer Divisor) „ample" ist im Sinne von Kodaira, einen Ausdruck für $H^0(V_n, D)=\dim|D|+1$ durch die rechte Seite von (2). Der gewaltige Fortschritt und die Klärung, welche der Hauptsatz für die algebraische Geometrie bedeutet, brauchen also nicht besonders betont zu werden.

Weitere Folgerungen aus dem Hauptsatz: (3) Betrachtet man das Bündel W der komplexen kovarianten p-Vektoren über V_n, so ergibt sich folgendes. $h^{p,q}$ sei der Rang des Raumes der harmonischen Differentialformen vom Typus (p, q) auf V_n und $\chi^y(V_n)=\sum_0^n (-1)^q h^{p,q}$. Es gilt, mit einer Unbestimmten y,

$$\sum_0^n \chi^p(V_n)y^p=\varkappa\Big[\prod\frac{-\gamma_i(y+1)}{e^{-\gamma_i(y+1)}-1}-\gamma_i y^i\Big].$$

Für $y=1$ ist die linke Seite gleich dem Index $\tau(V_n)$, und man findet das Index-Theorem $\tau(V_n)=\varkappa[\prod \gamma_i/\mathrm{tgh}\,\gamma_i]$ des Verf. wieder (loc. cit.), welches allgemeinere Gültigkeit hat.

Der Beweis des Hauptsatzes wird skizziert; er führt über die Spezialfälle (1)—(3) und benützt insbesondere das Index-Theorem des Verf., ferner Sätze von Kodaira und Spencer. Ein wesentlicher Schritt beim Beweis ist die topologische Deutung der formalen Wurzeln δ_i der Chernschen Klassen von W durch Uebergang zu einem Faserbündel, welches in C_1-Bündel zerfällt. Zum verwendeten Apparat gehören direkt oder indirekt viele wichtige Gebiete und Methoden der modernen algebraischen Topologie. *B. Eckmann* (Zürich).

See also: Kolchin, p. 455; Skolem, p. 464.

NUMERICAL ANALYSIS

★**Skan, Sylvia W. Handbook for computers. Two volumes.** National Physical Laboratory, Department of Scientific and Industrial Research, 1954. 318 pp. (mimeographed). 15s.

A complete introduction to the elementary mathematics and to some of the techniques of computation that are required of junior computers is clearly presented with many illustrative examples. These chapters would constitute the basis for an on-the-job training program for an audience with limited mathematical background. The mathematical topics range from the solution of quadratic equations to an introduction to the calculus. The numerical techniques that are described range from finding square roots to numerical differentiation and integration of functions. Throughout there is the necessary emphasis on the numerical checks and the arrangement of work sheets which insures accuracy in a computation laboratory. *E. Isaacson* (New York, N.Y.).

★**Cuénod, Michel. Méthode de calcul à l'aide de suites.** Thèse, Ecole Polytechnique Fédérale, Zurich, 1955. Imprimerie La Concorde, Lausanne, 1955. 75 pp.

In this thesis the author has developed, and applied to a wide variety of problems, the procedure suggested by A. Tustin [J. Inst. Elec. Engrs. IIA. **94** (1947), 152–160], in which functions of time are represented by the set of values of ordinates at equal intervals, such sequences of numbers being referred to as "time series". Convolution of two functions corresponds with the multiplication of polynomials whose coefficients are the terms of these series. The inverse process, corresponding with division of polynomials, is possible if the functions are zero up to an initial time. The author examines the elementary operations on and with time series, including integration and differentiation. He then shows the application to the approximate numerical solution of linear differential equations for such cases as arbitrarily varying forces or the deflection of an arbitrarily loaded elastic string. Cases of simple non-linear equations are solved. The propagation of disturbances along transmission lines or water-hammer in pipes are examples of problems in which the use of time series is shown to be convenient. Random functions are then considered, with special reference to the determination of the mean-square value of a variation linearly related to a random 'input'. The result corresponds with well-known relationships in terms

of correlation functions; and an example is given of the calculation of the mean-square fluctuation of speed of a prime mover resulting from a random variation of load, of which an adequate sample is given. A final section gives an analysis of the errors associated with approximation by time series. *A. Tustin* (London).

Cuénod, Michel. Principes du calcul à l'aide de suites. Bull. Tech. Suisse Romande **81**, 187–200 (1955).

This paper is identical with the earlier part of the thesis reviewed above. *A. Tustin* (London).

Andersen, Einar. Adjustment of observations by the method of least squares. Geodaet. Inst. Skr. (3) **22** (1955), 51 pp.

An elementary exposition in matrix terms, with numerical examples. *A. S. Householder.*

Aparo, Enzo. Risoluzione numerica di un problema di minimi quadrati. Ricerca Sci. **25** (1955), 3039–3044.

If $ap=c$, and $c^*c=d$ is nonsingular, then one verifies that $x=pd^{-1}c^*b$ satisfies $a^*ax=a^*b$. The author considers solving the normal equations by obtaining c with orthogonal columns and p a unit upper triangle. Hence one intends d to be diagonal. Through rounding errors, however, one obtains in fact a matrix $\tilde{c}$ for which $\tilde{c}^*\tilde{c}=d+e$ differs from being diagonal by a symmetric matrix e of small elements and null diagonal. This leads to an approximate $\tilde{x}=pd^{-1}\tilde{c}^*b$, but if elements of e are sufficiently small a more exact solution is given by $\tilde{x}+\xi$, where $\xi=-pd^{-1}ed^{-1}\tilde{c}^*b$. In particular, an improvement is assured when

$$\max \sum_k |e_{hk}| < \min d_{hh}.$$

In calculating p, one computes directly $q=p^{-1}$, column by column. The author notes, however, the formal identity of the relations $cq=a$, $pq=e$ (the identity), and obtains corresponding columns of q, c and p in that order. *A. S. Householder* (Oak Ridge, Tenn.).

Penrose, R. On best approximation solutions of linear matrix equations. Proc. Cambridge Philos. Soc. **52** (1956), 17–19.

The author recently [same Proc. **51** (1955), 406–413; MR **16**, 1082] introduced a unique generalized inverse $A^\dagger$ of a matrix A valid for rectangular and singular matrices. He calls X_0 a "best approximate solution" of the equation $f(X)=G$ if for all X, either $\|f(X)-G\|>\|f(X_0)-G\|$ or $\|f(X)-G\|=\|f(X_0)-G\|$ and $\|X\|>\|X_0\|$ where $\|X\|$ means the sum of the squares of the moduli of the elements of X. It is shown that $A^\dagger B$ is the unique best approximate solution of $AX=B$. The case $B=I$ suggests an alternative definition for $A^\dagger$. The case where X and B are vectors is the well known problem of finding best approximate solutions of systems of inconsistent equations. It is shown that the linear matrix equation $\sum A_iXB_i=B$ comes under this heading. Methods for the computation of $A^\dagger$ are suggested and linked up with the computation of the ordinary inverse of a matrix. *O. Taussky-Todd.*

Householder, A. S. Terminating and nonterminating iterations for solving linear systems. J. Soc. Indust. Appl. Math. **3** (1955), 67–72.

A new iterative procedure for solving a linear (not necessarily symmetric) system $Ax=h$ is developed. Proceeding from an approximation point x_r in a chosen direction u_{r+1} to the next approximation $x_{r+1}=x_r+\lambda_{r+1}u_{r+1}$, the scalar λ_{r+1} is determined in such a way that the residual $v_{r+1}=h-Ax_{r+1}$ is orthogonal to Au_{r+1}. From this follows that at each step the residual is diminished in length. Hence the iteration is convergent. If the directions u_r are chosen such that the vectors Au_r build an orthogonal system, the algorithm yields the exact solution after at most n steps, A being a n-row matrix. An orthogonalisation technique in order to perform this is established. A special version of the algorithm given in the paper coincides with the method of conjugate gradients [M. R. Hestenes and E. Stiefel, J. Res. Nat. Bur. Standards **49** (1953), 409–436; MR **15**, 651]. *E. Stiefel* (Zürich).

Roth, J. P. An application of algebraic topology to numerical analysis: on the existence of a solution to the network problem. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 518–521.

The author considers the following algebraic situation: (Reviewer's notation) Let A, B, C be finite-dimensional vector spaces over the complex numbers with homomorphisms: $0\to A\to B\to C\to 0$ (i.e. A is a subspace of B, and C the quotient space). In the dual spaces there is induced a corresponding sequence $0\leftarrow A^*\overset{j^*}{\leftarrow}B^*\leftarrow C^*\leftarrow 0$. Let L be an isomorphism of B onto B^*, and consider the following problem: Given $c\in C$, $a^*\in A^*$, there are to be found elements $b\in B$, $b^*\in B^*$ so that

$$b^*=Lb,\ ib=c,\ j^*b^*=a^*.$$

The author shows that this problem can always be solved and then uniquely, provided $L':A\to A^*$ induced by L, is an isomorphism.

By interpreting the steady-state network problem in this frame-work (A as 1-cycles of the network, B as 1-chains, C as reduced 0-chains, L-"the impedance matrix" of the various branches) the author generalizes the usual positive-definiteness condition on L for the existence and uniqueness of the network problem, to the condition $L+L^t$ positive definite. *R. Bott* (Princeton, N.J.).

Roth, J. P. The validity of Kron's method of tearing. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 599–600.

Using his earlier description of the network problem [see the paper reviewed above], the author applies it to Kron's method of tearing. The idea of this method is the following one: Given two networks K and $\tilde{K}$ with the same number and a simplicial map of K onto $\tilde{K}$ which induces an isomorphism on the space of branches, every network problem on K induces a network problem on $\tilde{K}$. Further, from a solution of the induced problem (which might be simpler) the original problem can be recaptured. *R. Bott* (Princeton, N.J.).

Kron, Gabriel. Solving highly complex elastic structures in easy stages. J. Appl. Mech. **22** (1955), 235–244.

The problem is to solve extensive systems of simultaneous linear algebraic equations describing physical structures. The author discusses his "method of tearing" [J. Appl. Phys. **24** (1953), 965–980; MR **15**, 747]. He contrasts it in numbers of machine operations with matrix partitioning methods, with which it has been wrongly identified by various persons. The ultimate purpose of the method of tearing is to enable large structures to be solved readily in terms of tabulated solutions of standard component structures. It is stated that the method of tearing is not intended to compete with existing methods of solving linear systems, when they work. The special

nature of elastic structures is discussed in terms of network analogies. Reviewer's note: Mathematicians interested in these matters should consult recent papers by J. P. Roth [for example, the paper reviewed above]. *G. E. Forsythe* (Los Angeles, Calif.).

Snyder, James N. On the improvement of the solutions to a set of simultaneous linear equations using the ILLIAC. Math. Tables Aids Comput. **9** (1955), 177–184.

de la Garza, A. Error bounds for a numerical solution of a recurring linear system. Quart. Appl. Math. **13** (1956), 453–456.

Let φ_0 be an approximate solution of $A\varphi=\eta$, and let $\varrho=\eta-A\varphi_0$, $\varepsilon=A^{-1}\varrho$. Suppose A irreducible, of the form $A=-m(I-D)$, where $m\geqq 0$ is a nonsingular diagonal matrix and $D\geqq 0$ has a null diagonal. Suppose, further, that $D\psi\leqq\psi$, but $D\psi\neq\psi$ where ψ is a column of ones. Then $(I-D)^{-1}=\sum_0^\infty D^\nu$ exists and is nonnegative. If $\alpha(M)$ represents the matrix whose elements are the moduli of those of M, and if $\alpha(\varrho)\leqq r_U\psi$, then

$$\alpha(\varepsilon)\leqq(I-D)^{-1}m^{-1}\alpha(\varrho)\leqq r_U(I-D)^{-1}m^{-1}\psi\leqq r_U m_L^{-1}(I-D)^{-1}\psi$$

where m_L^{-1} is the maximal element in m^{-1}. A fortiori, if $E\geqq D$ but $\sum_0^\infty E^\nu$ converges, then $(I-E)^{-1}\psi\geqq(I-D)^{-1}\psi$, whence if τ satisfies $(I-E)\tau=\psi$, then $\alpha(\varepsilon)\leqq r_U m_L^{-1}\tau$. Somewhat greater generality can be had by adjoining null rows and columns to D, nonnegative rows and columns to E.

Application is made to the simple finite-difference approximation to $\nabla^2\varphi=f\varphi+g$ in a region of R of the plane with $f\geqq 0$ and φ prescribed on the boundary of R. If S is a rectangle containing R in its interior and with sides parallel to the axes, one can set up difference equations in S of the form $(I-E)\tau=\psi$ and these can be solved by what corresponds to a separation of variables. This leads to an upper bound for the error in the solution of the difference equations. *A. S. Householder* (Oak Ridge, Tenn.).

Gregory, Robert T. On the convergence rate of an iterative process. Math. Mag. **29** (1955), 63-68.

This paper contains some remarks of an expository nature concerning the convergence of an iterative process. *O. Taussky-Todd* (Washington, D.C.).

Washizu, K. On the bounds of eigenvalues. Quart. J. Mech. Appl. Math. **8** (1955), 311–325.

"In this paper, formulae for locating the bounds of eigenvalues of a real symmetric matrix are treated in a unified manner. In the first part of the paper, several theorems which provide bounds for eigenvalues are represented in a geometrical space. In the second part, a generalization of Rayleigh's principle is attempted, and some theorems which provide bounds for eigenvalues are discussed as special cases of the generalized Rayleigh's principle. The system treated here has a finite number of degrees of freedom. By appropriate modifications the above principles can, however, be extended to a system with an infinite number of degrees of freedom."

E. Isaacson (New York, N.Y.).

★ **Tompkins, C. Projection methods in calculation.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 425–448. National Bureau of Standards, Washington, D.C., 1955.

The paper discusses some computational experience of the author and his coworkers with the solution of systems of equations or inequalities by cyclic projections.

The cyclic projection scheme is one designed for computational convenience. Each equation or inequality is considered individually. An approximate solution is "improved" by the following two-step process applied cyclically to the equalities and inequalities: 1. The gradient of the function which should be zero if the equality is satisfied (or which should be non-negative if the inequality is satisfied) is computed at the approximate solution; 2. The improved point is calculated along the line of the gradient through the approximate solution at a distance which would cause the relation in question to be satisfied if the rate of change along the gradient line were constant and equal to its value at the approximate solution.

This method is, in other words, an application of a Newton method of solution cyclically to the individual relations. For linear equalities it is due to Kaczmarz, and the process is straightforward projection onto hyperplanes. For quadratic equations, the improved point is half way between the earlier approximate solution and its projection onto its polar. The paper reports work by W. L. Hart, T. S. Motzkin, E. E. Osborne, David Pope, and the author. *T. L. Saaty* (Washington, D.C.).

Eisemann, Kurt. Linear programming. Quart. Appl. Math. **13** (1955), 209–232.

The "simplex" process, as a computational procedure for solving Linear Programming problems with and without degeneracy, is discussed. The ideas are essentially the author's competent version of the same material found in "An introduction to linear programming" [Wiley, New York, 1953; MR **15**, 48], by A. Charnes, W. W. Cooper and A. Henderson, and in the papers by G. B. Dantzig [pp. 339–347, 359–373; MR **15**, 47, 48] and Brown and Koopmans [pp. 377–380; MR **16**, 289] in "Activity analysis of production and allocation", edited by T. C. Koopmans [ibid., 1951]. *T. L. Saaty.*

Charnes, A., and Cooper, W. W. The stepping stone method of explaining linear programming calculations in transportation problems. Management Sci. **1** (1954), 49–69.

This paper presents an exposition of the simplex method for solving the transportation problem. Novel features include a new discussion of degeneracy and a procedure for finding all optimal basic solutions.

A. J. Hoffman (Washington, D.C.).

Charnes, A., and Lemke, C. E. Minimization of non-linear separable convex functionals. Naval Res. Logist. Quart. **1** (1954), 301–312 (1955).

The authors consider the problems of minimizing the sum of convex functions $f_i(x_i)$ defined on all points $(x_1, \cdots, x_n)$ satisfying a system of linear inequalities. Their technique consists of constructing a large linear programming problem whose solution yields the solution to a polygonal approximation of the convex programming problem. They also present an efficient algorithm for solving the auxiliary problem. *A. J. Hoffman.*

★ **Hoffman, A. J. How to solve a linear programming problem.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 397–424. National Bureau of Standards, Washington, D.C., 1955.

This paper is essentially concerned with computational

aspects of linear programming. A good discussion of devices for special situations arising in the formulation and the solution of a linear programming problem, such as the use of the dual, parametric minimization, and other possible simplifications, is given. An excellent bibliography of papers in the field is supplied.

T. L. Saaty (Takoma Park, Md.).

de Greiff Bravo, Luis. Application of the theory of finite differences to the calculation of polynomials. Rev. Mat. Elem. 4 (1955), 1–12. (Spanish)

Löffler, E. Die Methode der "ersten Zeile", ein neues Verfahren zur Bestimmung der komplexen Lösungen von Polynom-Gleichungen höheren Grades. Z. Angew. Math. Mech. 35 (1955), 434–437.

An algorithm is set up for the computation of the complex roots of a polynomial equation with real coefficients when the real roots and the moduli of the complex roots are given. The method is carried through with formulas based on relations between certain product sums of the real roots and the moduli and real parts of the complex roots with the coefficients of the given equation.

E. Frank (Chicago, Ill.).

Ludwig, Rudolf. Ein praktisches Verfahren zur Lösung biquadratischer Gleichungen mit nur komplexen Wurzeln. Z. Angew. Math. Mech. 35 (1955), 401–405. (English, French and Russian summaries)

In this paper a very useful method for the computation of the complex roots of a polynomial equation with real or complex coefficients is described. The method consists of the separation of the equation into the real and imaginary parts. The resultant of the two new equations thus formed has the real parts of the original equation as roots. A method of successive approximations for the computation of the real roots of the resultant is given. A simple formula is shown for the corresponding imaginary parts.

E. Frank (Chicago, Ill.).

Demčenko, O. P. Obtaining the frequency characteristics of systems of automatic regulation by means of Mihallov's curves. Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 693–696. (Russian)

The steady-state solution $x_\infty(t)$ of the equation

$$(a_0D^n+\cdots+a_n)x=(b_0D^m+\cdots+b_m)\sin\omega t$$

is constructed graphically.

M. Golomb.

Ho, Kuo-Chu. Double interpolation formulae and partial derivatives in terms of finite differences. Math. Tables Aids Comput. 9 (1955), 52–62.

The main part of this paper is devoted to a tabulation of partial derivatives in terms of finite differences. In table I the different types of double interpolation formulae for the nine several regions of operation are classified into five types (double forward, semi-forward, double backward, semi-central and double-central). In table II the partial derivatives of the first four orders are expressed by means of double forward, semi-forward and double backward differences. In table III the same by semi-central differences (Stirling and Bessel), in table IV the same by means of central differences. The expressions in terms of f_{ij} of the partial derivatives up to the third order are written out explicitly for all five types in Tables V and VI.

S. C. van Veen (Delft).

Romberg, Werner. Vereinfachte numerische Integration. Norske Vid. Selsk. Forh., Trondheim 28 (1955), 30–36.

It is known how to approximate an integral $I=\int_0^2 F(x)dx$ by a weighted sum $I_n=\sum_{k=0}^{n} p_k^{(n)}F(kh)$ where $h=1/n$. The weights p_k can be chosen such that $I=I_n$ for any polynomial F of degree n. On the other hand it is known how to linearly combine two trapezoidal formulas related to n and $2n$ as so to obtain higher accuracy. The author iterates this latter procedure in order to attain higher and higher accuracy. He claims that this way involves less computational work than the application of a formula I_n with the same order of error. Some numerical examples are given.

H. Bückner (Schenectady, N.Y.).

Salzer, Herbert E. Osculatory quadrature formulas. J. Math. Phys. 34 (1955), 103–112.

An "osculatory quadrature formula" is an integration formula which utilizes both the function (integrand) and its first derivatives at n fixed points. It may be obtained by integrating for each n Hermite's n-point osculatory interpolation formula. By setting:

$$x=x_0+ph,\ f(x)=f(x_0+ph)=f_p=f;$$
$$x_i=x_0+ih,\ f(x_i)=f_i,\ f'(x_i)=f_i'.$$

Hermite's n-point osculatory formula for equally spaced arguments at intervals of h becomes

$$f(x_0+ph)=\sum_{i=-[(n-1)/2]}^{[n/2]}\{L_i^{(n)}(p)\}^2\{1-2L_i^{(n)\prime}(i)(p-i)\}f_i+\sum_{i=-[(n-1)/2]}^{[n/2]}\{L_i^{(n)}(p)\}^2(p-i)hf_i'+R_{2n}(p),$$

where

$$L_i^{(n)}(p)=\prod_{j=-[(n-1)/2]}^{[n/2]}\{(p-j)/(i-j)\}\quad (j=i \text{ absent}),$$

$$R_{2n}(p)=f^{(2n)}(\xi)h^{2n}\Big\{\prod_{j=-[(n-1)/2]}^{[n/2]}(p-j)\Big\}^2/2n!$$
$$=f^{(2n)}(\xi)h^{2n}M^{(n)}(p);$$
$$x_{-[(n-1)/2]}\leqq\xi\leqq x_{[n/2]}.$$

By integration the osculatory quadrature formulae are obtained, viz

$$\int_{x_0+sh}^{x_0+rh}f(x)dx=h\sum_{i=-[(n-1)/2]}^{[n/2]}A_if_i+h^2\sum_{i=-[(n-1)/2]}^{[n/2]}B_if_i'+R.$$

The author gives tables of the exact fractional values, in lowest terms, of the coefficients A_i and B_i for n ranging from two to seven, and for all values of r and s each within the range $(-[(n-1)/2], [n/2])$. These tables are supplemented by a schedule tabulating the exact values of the remainder term

$$R=\int_{x_0+rh}^{x_0+sh}R_{2n}(x)dx=h^{2n+1}f^{(2n)}(\xi(\eta))\int_r^s M^{(n)}(p)dp;\ r\leqq\eta\leqq s;$$
$$f^{(2n)}(\xi(\eta))=f^{(2n)}(\xi_1);\ x_{-[(n-1)/2]}\leqq\xi_1\leqq x_{[n/2]}.$$

As a typical example we give the case $r=0$, $s=1$, $n=3$:

$$\int_{x_0}^{x_0+h}f(x)dx=h\{\tfrac{11}{240}f(x_0-h)+\tfrac{1}{15}f(x_0)+\tfrac{101}{240}f(x_0+h)\}+$$
$$h^2\{\tfrac{1}{30}f'(x_0+h)+\tfrac{1}{15}f'(x_0)-\tfrac{13}{240}f'(x_0+h)\}+\tfrac{1}{5450}h^7f^{(6)}(\xi);$$
$$x_0-h\leqq\xi\leqq x_0+h.$$

S. C. van Veen (Delft).

Adachi, Ryuzo. On the numerical solution of the simultaneous differential equations under some conditions. II. Kumamoto J. Sci. Ser. A. **2** (1954), no. 1, 24–39.

In earlier papers [same J. **1** (1954), no. 3, 14–33, no. 4, 28–30; MR **16**, 631] the author has treated simultaneous ordinary differential equations where the solutions must satisfy certain conditions at the intersections with given loci. In the present paper a number of fully worked examples are presented to illustrate several types of cases which may occur. The remainder of the paper is devoted to the question of finding practical tests for convergence of the process, a question not yet answered to the author's complete satisfaction. *W. E. Milne.*

Adachi, Ryuzo. A method on numerical solution of some differential equation. Kumamoto J. Sci. Ser. A. **2** (1954), no. 1, 40–46.

The differential equation $L(x)+f(x, y)=0$ in which $L(x)$ is a linear second-order differential expression of Sturm-Liouville type can be replaced by integral equation

$$y(x)=\int_a^b K(x, \xi)f(\xi, y(\xi))d\xi.$$

The author proposes to replace the integral equation by a finite number of algebraic equations in the well-known manner. Since these are not in general linear an iterative method may be tried for their solution. This is applied in detail to the example $y''+ay-by^3=0$. *W. E. Milne.*

Bachellier, J., et Gardy, H. Application du calcul symbolique graphique à l'étude des systèmes différentiels linéaires. Ann. Fac. Sci. Univ. Toulouse (4) **18** (1954), 161–177 (1955).

The authors consider the experimental analysis of linear control systems by means of measuring an input function $F(t)$ and a response $f(t)$ with t denoting the time. The emphasis is on chemical process control where in general large time constants (order of 30 min or even greater) do not permit one to apply special functions $F(t)$ like step functions or sinusoidal oscillations. For this reason it becomes necessary to measure functions $F(t)$ and $f(t)$ as they naturally appear and to evaluate the function $U'(i\omega)=\varphi(i\omega)/\Phi(i\omega)$ with $\Phi(i\omega)=\int_0^\infty F(t)e^{-i\omega t}dt$ and $\varphi(i\omega)=\int_0^\infty f(t)e^{-i\omega t}dt$. In order to evaluate the integrals φ, Φ which are assumed to exist for $\omega\geqq 0$, a decomposition to finite integration limits is made as well as a reduction of the components to the typical form $G(\theta_0, \theta_1)=\int_{\theta_0}^{\theta_1} g(\theta)e^{-i\theta}d\theta$. The evaluation of G is carried out graphically in the complex plane; G is represented approximately by a weighted sum of ordinates $g(\theta)$ for the abscissae $\theta=\theta_0$, $\frac{1}{3}(2\theta_0+\theta_1)$, $\frac{1}{3}(\theta_0+2\theta_1)$, θ_1. (While the weights follow clearly from the description of the graphical method, there seems to be no indication of what θ_0 and θ_1 are supposed to be; these abscissae cannot be independent of the weights.) Some special advice is given for dealing with extremely small and extremely large frequencies ω. Two examples, one of which comes from an experiment, indicate the applicability of the graphical method within the order of 1% of error. *H. Bückner.*

Tatarkiewicz, K. Une méthode d'estimation de l'erreur dans le procédé de Ritz. Ann. Polon. Math. **1** (1955), 346–359.

Let $y_0(x)$ be the unique solution of $(py')'-qy=f$; $y(0)=y(1)=0$, where $p(x)\geqq\bar{p}>0$, $q(x)\geqq 0$, $p\in C^1$ and $q\in C$ on $0\leqq x\leqq 1$. Let $y_n(x)$ be an approximation to $y_0(x)$ obtained by the Ritz process with $y_n(0)=y_n(1)=0$. The author shows that the error $r_n=\max|y_n-y_0|$ on $0\leqq x\leqq 1$, satisfies $r_n\leqq KM_n$, where $M_n=\int_0^1|(py_n')'-qy_n-f|\,dx$, and K is a constant. Various estimates for K are derived. Simpler estimates are obtained when the additional hypothesis $q(x)\geqq\bar{q}>0$ is imposed. *W. S. Loud.*

Pöschl, Th. Über eine mögliche Verbesserung der Ritzschen Methode. Ing.-Arch. **23** (1955), 365–372.

The author approximates the solution of the boundary-value problem for a differential equation that arises from a variational problem. He proposes that the Ritz method for solving the variational problem be combined with the collocation requirement. That is, the admissible functions $\{w_i\}$, which are selected to satisfy appropriate boundary conditions and to be a complete family are, in addition, required to satisfy the differential equation at a finite number of points. The solution of the original problem is then approximated by the linear combination of the $\{w_i\}$ that minimizes the variational integral. *E. Isaacson.*

Volkov, E. A. On numerical solution of the problem of Lavrent'ev-Bicadze. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 755–758. (Russian)

The author exhibits a difference method for the upper half-plane in approximating the solution of the Lavrent'ev-Bicadze equation, differing somewhat from that of Ladyženskaya [Uspehi Mat. Nauk (N.S.) **9** (1954), no. 4(62), 187–189; MR **16**, 1120]. *W. E. Milne.*

Vernotte, Pierre. Généralisation d'un procédé d'intégration pratique des équations aux dérivées partielles. Application à la diffusion de la matière ou de la chaleur. C. R. Acad. Sci. Paris **241** (1955), 1699–1700.

Chu, Chiao-Min, and Churchill, Stuart W. Numerical solution of problems in multiple scattering of electromagnetic radiation. J. Phys. Chem. **59** (1955), 855–863.

"The steady-state scattering and absorption of electromagnetic radiation by dense dispersions of particles was investigated theoretically. The results should also have applicability in describing the behavior of thermal neutrons in dense material. Particular attention was given to dispersions of finite thickness and to anisotropic distribution of scattered energy characteristic of wave lengths less than the circumference of the particles. The objective was to find methods of solution suitable for numerical calculations. Existing analytical solutions of problems in multiple-scattering were found to be limited to isotropic scattering and/or dispersions of infinite extent. Approximate and numerical methods of solution proposed by previous investigators were evaluated with respect to accuracy and usability. A six-flux model was developed which reduces the general transport equation to a set of simple differential equations. Solutions suitable for numerical calculation can be obtained for most geometrical configurations of source and dispersion. An analytical solution is presented for an obliquely incident wave and a parallel-plane dispersion of finite thickness. Extension of the results to point and plane sources outside a dispersion is outlined. Degeneration of the six-flux model into a very simple two-flux model is also considered. The transmission through representative dispersions was calculated using the six-flux model, the simple two-flux model, and the diffusion equation, and the results are compared graphically." *E. Isaacson* (New York, N.Y.).

★ **Warschawski, S. E. On the solution of the Lichtenstein-Gershgorin integral equation in conformal mapping. I. Theory.** Experiments in the computation of conformal maps, pp. 7–29. National Bureau of Standards Applied Mathematics Series, No. 42. U. S. Government Printing Office, Washington, D. C., 1955. 40 cents.

Let C be a closed Jordan curve with continuously turning tangent represented by the equation $z=\zeta(s)$, where the arc length s is the parameter, $0\leq s\leq L$. Suppose that $w=f(z)$ maps the interior R of C conformally onto the circle $|w|<1$ so that $z=z_1$ in R and $z=z_2$ on C correspond to $w=0$ and $w=1$, respectively. Then $f(z)$ can be extended to C so that $f(z)$ becomes continuous in the closure of R. The function $\theta(s)=\arg f(\zeta(s))$ satisfies the integral equation

$$\theta(s)=\int_0^L K(s,t)\theta(t)dt-2\beta(s).$$

Here $\beta(s)=\arg[(z_2-\zeta(s))/(z_1-\zeta(s))]$,

$$K(s,t)=\pi^{-1}\cdot\sin(\tau-\varphi)/r_{st},$$

where $\tau=\tau(t)$ is the tangent angle, $\varphi=\arg[\zeta(t)-\zeta(s)]$, $r_{st}=|\zeta(s)-\zeta(t)|$. The solution of the integral equation by iteration is studied. For very general classes of regions, explicit estimates of the error after n iterations both for $\theta(s)$ and for its first derivative $\theta'(s)$ are obtained in terms of $1/|\lambda_2|$, where λ_2 is that eigenvalue of $K(s,t)$ whose absolute value is nearest to 1. This leads to appraisals for $1/|\lambda_2|$ and upper bounds are obtained for a wide class of regions. The estimates are of the following type. Let C_0 be a simple closed curve, $K_0(s,t)$ the kernel pertaining to C_0, and Λ_2 the second characteristic value of $K_0(s,t)$, supposed to be known. If C_0 is close to C (in a certain specified sense), then a bound for $1/|\lambda_2|$ is obtained in terms of $1/|\Lambda_2|$. Other kinds of estimates are obtained which are applicable to convex and certain other related types of regions. *W. Seidel.*

★ **Todd, John, and Warschawski, S. E. On the solution of the Lichtenstein-Gershgorin integral equation in conformal mapping. II. Computational experiments.** Experiments in the computation of conformal maps, pp. 31–44. National Bureau of Standards Applied Mathematics Series, No. 42. U. S. Government Printing Office, Washington, D. C., 1955. 40 cents.

The process described in the previous paper was tested by carrying out the iterations on an automatic computer in the cases of ellipses of varying shapes. The cases $k=1.2$, $k=2$, $k=5$ are examined in detail, where the ellipse is given in the parametric form $z=a\cos t$, $y=b\sin t$, $0\leq t<2\pi$, and $k=a/b$. It is found that the results obtained are in close agreement with the theoretical estimates of the previous paper. Tables are appended at the end.

W. Seidel (Notre Dame, Ind.).

★ **Ostrowski, A. M. Conformal mapping of a special ellipse on the unit circle.** Experiments in the computation of conformal maps, pp. 1–2. National Bureau of Standards Applied Mathematics Series, No. 42. U. S. Government Printing Office, Washington, D. C., 1955. 40 cents.

In order to compare the efficiency of various numerical methods for the computation of conformal maps, the author computes the arguments of the points on the ellipse $\xi^2/a^2+\eta^2/b^2=1$, $a=1.2$, $b=1$, corresponding to the points $z_\nu=\exp(i\nu\pi/40)$ $(\nu=0, 1, \cdots, 20)$ on the unit circle $|z|=1$ under the conformal mapping $\zeta=f(z)$, $f(0)=0$, $f'(0)>0$, of the ellipse on the circle. The explicit mapping function is used and the result is calculated with an error less than 10^{-7}. A table is appended. *W. Seidel.*

★ **Ostrowski, A. M. Theodorsen's and Garrick's method for conformal mapping of the unit circle into an ellipse.** Experiments in the computation of conformal maps, pp. 3–5. National Bureau of Standards Applied Mathematics Series, No. 42. U. S. Government Printing Office, Washington, D. C., 1955. 40 cents.

The problem of the paper reviewed above is solved by the Theodorsen-Garrick method. Denoting by $\varrho(\varphi)\exp\{i\theta(\varphi)\}$ the point on the ellipse corresponding to the point $e^{i\varphi}$ on the unit circle, the conformal mapping can be determined from the integral equation

$$\theta(\varphi)-\varphi=-\frac{1}{2\pi}\int_0^\pi[P(\theta(\varphi+t))-P(\theta(\varphi-t))]\cot\tfrac{1}{2}t\,dt,$$

where $P(\theta)=\log\varrho(\varphi)$. In an earlier paper [Construction and applications of conformal maps, Nat. Bur. Standards Appl. Math. Ser. No. **18** (1952), 165–174; MR **14**, 909], the author replaced the integral equation by an approximate finite system of simultaneous equations. In the present numerical example the circle is decomposed into 20 equal parts. Since the map is doubly symmetric, the problem reduces to that of solving a system of 4 equations in 4 unknowns. The system is first solved by Newton's method with an absolute error less than 9×10^{-8} and the differences between these values and the "true values" obtained in the previous paper are noted. Secondly, the same system is solved by iteration and a table of the iterates is given. It is found that the third iterate gives values with an error less than 1.3×10^{-4}. *W. Seidel.*

Štykan, A. B. Graphical solution of integro-differential equations. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 171–180. (Russian)

Graphical solutions are constructed for integro-differential equations of the form

$$u'(x)=f(x)+\xi\Big[\int_0^a K(x,s)\mu[u(s)]ds\Big]$$

or of the form

$$u(x)=f(x)+\xi\Big[\int_0^a K(x,s)\mu[u'(s)]ds\Big],$$

in which the $K(x,s)$ is of the special type $K(x,s)=\psi(x)/\Phi(s)$. The process is then generalized to apply where $u(x)$ above is replaced by $u^{(m)}(x)$ and $u'(s)$ is replaced by $u^{(n)}(s)$. *W. E. Milne* (Corvallis, Ore.).

Lyapunov, A. A. On a criterion for verification of the interpretation of gravitational observations. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 265–266. (Russian)

Somtimes the interpretation of gravitational maps proceeds by a trial and error method. In such a case it is necessary to check the hypothetical shape S of disturbing masses with the aid of an identity which correlates the mapped quantity with S. The author considers the two-dimensional problem and proves the identity

$$\int_a^b g(x)dx=k\cdot d\iint_\sigma\varphi(\xi,\eta)d\xi d\eta,$$

where k is the gravitational constant, d the density contrast and $\varphi(\xi,\eta)$ is the angle under which the segment

(a,b) of the typical profile is seen from a point (ζ,η) interior to the normal section σ of the cylindrical disturbing masses. This identity can be used for various values of a and b on the typical profile to check the hypothetical shape S of the normal section σ. *E. Kogbetliantz.*

Roth-Desmeules, Ernst. Zur Berechnung der Geschossabweichung unter dem Einfluss eines Seitenwindes. Z. Angew. Math. Phys. **6** (1955), 494–497.

See also: Laasonen, p. 489; Koiter, p. 498; Krook, p. 543.

Tables

★**Table of hyperbolic sines and cosines, $x=2$ to $x=10$.** National Bureau of Standards Applied Mathematics Series, no. 45. U.S. Government Printing Office, Washington, D.C., 1955. v+81 pp. 55 cents.

An earlier publication [same Series, no. 36 (1953); MR **15**, 352] gave nine decimal values of $\sinh x$ and $\cosh x$ for $x=0.0(0.0001)2.0$ and $x=0.0(0.1)10.0$. The present tables give nine significant figures for $x=2.0(0.001)10.0$, thus facilitating interpolation in the extended range. For $x>10$ both functions are represented to nine significant figures by $\frac{1}{2}e^x$. *L. Fox* (Teddington).

★**Table of the descending exponential, $x=2.5$ to $x=10$.** National Bureau of Standards Applied Mathematics Series, no. 46. U.S. Government Printing Office, Washington, D.C., 1955. v+76 pp. 50 cents.

These twenty-decimal tables of e^{-x} cover the range $x=2.5(0.001)10.0$, and are an extension of previous tables [Nat. Bur. Standards Appl. Math. Ser. no. 14 (1951); MR **13**, 386] which gave eighteen decimals for $x=0.0(0.0001)2.5$. Interpolation is conveniently performed with the use of the addition formula and certain auxiliary tables given in the earlier volume. The calculations were performed on the NBS computer SEAC, checked by differencing on a punched-card calculator, and printed from a punched-card controlled calculator. *L. Fox.*

★**Stratton, J. A., Morse, P. M., Chu, L. J., Little, J. D. C., and Corbató, F. J. Spheroidal wave functions, including tables of separation constants and coefficients.** The Technology Press of M.I.T. and John Wiley & Sons, Inc., New York; Chapman and Hall, Ltd., London, 1956. xiii+613 pp. $12.50.

This volume is a successor of (and replacement for) "Elliptic cylinder and spheroidal wave functions" [Wiley, New York, 1941; MR **4**, 281] by Stratton, Morse, Chu, and Hutner. Since the appearance of the older work, elaborate tables relating to Mathieu functions became available [Nat. Bur. Standards, Columbia Univ. Press, 1951; MR **12**, 859]. For this reason the authors omitted Mathieu functions completely, and expanded the tables on spheroidal wave functions to a remarkable extent. "The present tables are the result of a number of years of effort on the part of the authors. Many persons at the Massachusetts Institute of Technology have contributed to the work···. It is to be hoped that the present volume will begin to make it possible for the applied physicist and engineer to handle wave problems in spheroidal coordinates with approximately the same degree of facility as has been possible previously for rectangular, circular cylindrical, and spherical coordinates.··· the present tables were computed, tabulated, and printed automatically···. One can thus expect a considerable reduction in errors···." (From the Foreword by Morse.) The tables have been produced on the Whirlwind I digital computer at the Massachusetts Institute of Technology.

The Foreword by Morse is followed by the reprint of a paper by Chu and Stratton [J. Math. Phys. **20** (1941) 259–309; MR **3**, 116]; an Introduction to the tables by Little and Corbató giving an account of the functions tabulated, the notations used, and the computation of the tables; and finally by some 550 p. of numerical tables.

The notation used is that of Morse and Feshbach [Methods of theoretical physics, 2 vols., McGraw-Hill, New York, 1953; MR **15**, 583] and differs from that of Stratton and Chu. The book by Meixner and Schäfke [Mathieusche Funktionen and Sphäroidfunktionen, Springer, Berlin, 1954; MR **16**, 586] reached the authors too late for a comparison of notations to be included in the present volume. The lack of a standard notation is regrettable, especially so since the best book on the subject, and the most extensive numerical tables differ on the point.

The differential equation for prolate spheroidal wave functions is

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+\left(A-h^2x^2-\frac{m^2}{1-x^2}\right)y=0;$$

for oblate spheroidal wave functions change x, h into $-ix$, ig. Angular wave functions are denoted by $S_{ml}(h,x)$, this being the function belonging to the lth characteristic value

$$A_{ml}(h)=l(l+1)+h^2\left[\frac{2l(l+1)-2m^2-1}{(2l-1)(2l+3)}+t_{ml}(h)\right];$$

and radial functions are denoted by $\mathrm{je}_{ml}(h,r)$. The functions are so normalized that S_{ml} behaves like $P_l^m(x)$ near $x=1$, and je_{ml} behaves like the spherical Bessel function $j_l(hr)$ as $r\to\infty$. The tables give numerical values of the separation constants, or rather of t, and of the coefficients in the Legendre function expansion of angular wave functions and in the Bessel functions expansions of radial wave functions, both for prolate and oblate spheroids and for $m=0(1)8$, $l=m(1)8$, g, $h=0(.1)1(.2)8$.

"The accuracy of the tables has been checked by the hand calculation of a number of cases. For the most part, all the significant figures published are good except for an occasional error of one in the last place. When the coefficient itself gets large ($\sim 10^8$), the last place may be weak. At no time is it thought that the numbers are off more than 5 in the last place" (From the Introduction to the tables.)

A very useful set of tables lavishly produced.

A. Erdélyi (Pasadena, Calif.).

Dank, M., and Barber, S. W. The specific heat function for a two-dimensional continuum. Numerical values of

$$\frac{C_2}{C_\infty}=\frac{6}{x^2}\int_0^x\frac{\xi^2d\xi}{e^\xi-1}-\frac{2x}{e^x-1}.$$

Math. Tables Aids Comput. **9** (1955), 191–194.

Computed values of C_2/C_∞ (see title) and C_2 are tabulated for $x=0(0.1)10(0.5)16$, 5D. Here $C_\infty=5.9616$.

Ishiguro, Eiichi, Yuasa, Sayoko, Sakamoto, Michiko, Kodaira, Chiyoko, and Higuchi, Sigeko. Tables useful for the calculation of the molecular integrals. VII. Nat. Sci. Rep. Ochanomizu Univ. **5** (1955), 197–212.

This instalment contains 15 p. of numerical tables of the integral $W_\tau^\nu(m, n; \alpha, \beta)$ defined in a previous review [Ishiguro, same Rep. **4** (1953), 64–76; MR **15**, 255].

A. Erdélyi (Pasadena, Calif.).

See also: Davis, p. 484; Hodges, p. 503; Kamat, p. 503; Todd and Warschawski, p. 540.

Mathematical Machines

Glossary of terms in the field of computers and automation. Computers and Automation **5** (1956), no. 1, 15–31.

Potters, M. L. A punched-card set-up for linear programming. Math. Centrum, Amsterdam, Rekenafdeling. Rep. MR **22** (1956), 17 pp.

Setterwall, M. A mechanical binary-decimal converter. J. Sci. Instrum. **33** (1956), 18–19.

Neovius, Gösta. Artificial traffic trials using digital computers. Ericsson Technics **11** (1955), 279–291.

Earnshaw, J. B. The diode pump integrator. Electronic Engrg. **28** (1956), 26–30.

Spring, Osc. W. Die maschinelle Berechnung der Erneuerungsfunktion. Mitt. Verein. Schweiz. Versich.-Math. **55** (1955), 417–422.

Cole, K. S., Antosiewicz, H. A., and Rabinowitz, P. Automatic computation of nerve excitation. J. Soc. Indust. Appl. Math. **3** (1955), 153–172.

Helmick, H. H., and Helmick, P. S. An electronic analog computer for the general physics laboratory. Proc. Iowa Acad. Sci. **62** (1955), 393–398.

Wentzel, Viggo. An electronic generator for functions of two independent variables. Ericsson Technics **11** (1955) 183–225.

Fuchs, H. Some aspects of the design of a D.C. amplifier for use with a slow analogue computer. Electronic Engrg. **28** (1956), 22–25.

Karplus, W. J. The use of electronic analogue computers with resistance network analogues. British J. Appl. Phys. **6** (1955), 356–357.

Atkinson, Cyril. Polynomial root solving on the electronic differential analyser (a technique for finding the real and complex roots of a polynomial using an electronic differential analyser). Math. Tables Aids Comput. **9** (1955), 139–143.

Èterman, I. L., and Obuvalin, M. I. A method of solving characteristic equations on electromodelling instruments. Avtomat. i Telemeh. **16** (1955), 554–555. (Russian)

Murray, C. T., and Hollway, D. L. A simple equipment for solving potential and other field problems. J. Sci. Instrum. **32** (1955), 481–483.

Harries, J. H. Owen. The rubber membrane and resistance paper analogies. Proc. I.R.E. **44** (1956), 236–248.

Gross, William A. A network representing elastic bodies in spherical coordinates. Proc. Iowa Acad. Sci. **62** (1955), 373–379.

Artobolevskiĭ, I. I. Theory of mechanisms for drawing curves which are hyperbolisms of conic sections. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 11, 62–72. (Russian)

Artobolevskiĭ, I. I. Theory of mechanisms for drawing cissoidal curves. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 12, 58–64. (Russian)

Artobolevskiĭ, I. I. On an inversor mechanism. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 825–827. (Russian)

See also: Parodi, p. 450; Snyder, p. 537; Nowinski and Turski, p. 557; Piesch, p. 564.

ASTRONOMY

Pylarinos, O. On the problem of n bodies. Bull. Soc. Math. Grèce **25** (1951), 49–67. (Greek)

In the classical problem of three bodies, Lagrange obtained explicit solution of the equations of motion for the case where the mutual distances of the three particles remain invariant in time. Under this condition the motion takes place in a plane and the particles describe circular orbits about their center of mass. Such motions are possible, if (i) the system is collinear or, if (ii) the lines joining the three particles form an equilateral triangle. In this paper the author has extended the Lagrange case to a system of n bodies by showing that (i) and (ii) are also true for n-particle systems and where the mutual interaction between them varies inversely as any power μ of the distance except $\mu=3$. *N. Chako.*

Sklyanskiĭ, A. L. On the theory of pairwise collision in the generalized three-body problem. Ukrain. Mat. Ž. **7** (1955), 160–166. (Russian)

Yu. D. Sokolov [same Ž. **2** (1950), no. 4, 18–24; MR 13, 996] discussed the triple collision in the rectilinear motion of three particles, mutually attracting under the action of forces with moduli

$$(1) \qquad g^2 m_i m_j e^{a/r_k} \quad (i, j, k=0, 1, 2;\ i\neq j\neq k),$$

where r_k denotes the distance between the particles P_i and P_j, g^2 and a being two positive constants. In the present paper the existence of trajectories of collisions between pairs of the particles is showed, assuming that the interactions are subject to the same law (1) and the motion is in space. *E. Leimanis* (Vancouver, B.C.).

Agostinelli, Cataldo. Soluzioni stazionarie delle equazioni della magneto idrodinamica interessanti la Cosmogonia. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1954), 216–221.

It is shown that the equations of hydromagnetics (appropriate for an inviscid incompressible fluid of finite

electrical conductivity) allow the stationary solution

$$\mathbf{H}=H_0\mathbf{k}+\gamma_0\lambda_0\mathbf{v},\quad \mathbf{v}=\boldsymbol{\omega}\times\mathbf{r},\quad \boldsymbol{\omega}=\mathbf{k}\omega,$$

where $\mathbf{k}$ is a unit vector in the z-direction (say), γ_0 and ω are arbitrary constants and $\lambda_0=\pm(4\pi\varrho/\mu)^{\frac{1}{2}}$. The corresponding pressure (p) distribution is given by

$$\frac{p}{\varrho}-U-(\tfrac{1}{2}-\gamma_0^2)\omega^2(x^2+y^2)=\text{const},$$

where U denotes the gravitational potential.

S. Chandrasekhar (Williams Bay, Wis.).

Agostinelli, Cataldo. Oscillazioni magneto idrodinamiche in una massa fluida rotante di dimensioni cosmiche, di forma ellissoidale rotonda. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89** (1955), 41–58.

In the paper reviewed above the author has shown that the equations governing an inviscid incompressible fluid of finite electrical conductivity allows a stationary solution (for the magnetic field, $\mathbf{H}$, and the velocity, $\mathbf{v}$) of the form

$$\mathbf{H}=H_0\mathbf{k}+\gamma_0\lambda_0\mathbf{v} \text{ and } \mathbf{v}=\omega\mathbf{k}\times\mathbf{r}$$

where $\mathbf{k}$ is a unit vector in the z-direction (say), $\lambda_0=\pm(4\pi\varrho/\mu)^{\frac{1}{2}}$ and γ_0 and ω are arbitrary constants. In this paper the author investigates the nature of the oscillations about this equilibrium state. The perturbation, h_z, in the z-component of $\mathbf{H}$ is shown to satisfy the equation (in cylindrical polar coordinates, z, φ)

$$(*)\qquad \Big(\frac{1}{\omega}\frac{\partial}{\partial t}+\frac{\partial}{\partial\varphi}\Big)^2h_z-\gamma_0^2\Big(\frac{\partial}{\partial\varphi}+\frac{v_0}{\omega}\frac{\partial}{\partial z}\Big)^2h_z$$
$$-\frac{k_0^2}{\omega}\Big(\frac{1}{\omega}\frac{\partial}{\partial t}+\frac{\partial}{\partial\varphi}\Big)\Delta_2h_z=0$$

where $v_0=H_0/\gamma_0\lambda_0$, $k_0^2=c^2/4\pi\mu\sigma$ and Δ_2 is the Laplacian. Equation (*) is separable in all three variables and fundamental solutions expressed in terms of Bessel functions of integral orders are found.

S. Chandrasekhar (Williams Bay, Wis.).

Agostinelli, Cataldo. Oscillazioni magneto idrodinamiche in una massa fluida cosmica uniformemente rotante dotata di un campo magnetico assiale e di un campo magnetico assiale e di un campo magnetico equatoriale rotante. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89** (1955), 68–92.

The analysis of the preceding paper is extended to the case when the initial magnetic field is of the somewhat more general form

$$\mathbf{H}=H_0'\mathbf{k}+H_0''e^{i\omega t}\mathbf{i}+\gamma_0\lambda_0(\mathbf{v}-\mathbf{v}_0),\quad \mathbf{v}=\mathbf{v}_0+\omega\mathbf{k}\times\mathbf{r}$$

where $\mathbf{i}$ is a unit vector at right angles to $\mathbf{k}$ and $\mathbf{v}_0$ is a uniform velocity. (The notation is the same as in the preceding review.) Oscillations about this equilibrium state are considered and the corresponding linearized equations are formally solved by expanding the various quantities in terms of a suitable orthonormal set constructed out of Bessel functions. *S. Chandrasekhar.*

Malkevič, M. S. On the vertical distribution of temperature in the atmosphere. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1955**, 166–177. (Russian)

L'auteur étudie la distribution verticale théorique de la température dans l'atmosphère en utilisant le mélange vertical et l'échange radiatif. La substance absorbante est la vapeur d'eau dont la distribution verticale est fournie par les observations. L'auteur calcule l'absorption sélective de la radiation par la vapeur d'eau. On trouve que le gradient vertical de la température est donné par une équation intégrale. La solution de cette équation nous donne la température de la couche supérieure de l'atmosphère. *M. Kiveliovitch* (Paris).

Krook, Max. On the solution of equations of transfer. I. Astrophys. J. **122** (1955), 488–497.

The equation of transfer appropriate for a conservative isotropically scattering semi-infinite atmosphere is considered. In view of the boundary condition at $\tau=0$, namely, $I(0,\mu)=0$ for $0>\mu\geqq-1$, the author introduces the moments

$$\alpha_k=\tfrac{1}{2}\int_0^1\mu^kI(\tau,\mu)d\mu \text{ and } \beta_k=\tfrac{1}{2}\int_{-1}^0\mu^kI(\tau,\mu)d\mu.$$

The equation of transfer leads to an infinite system of linear differential equations with constant coefficients for these moments and the author shows how the results derived from the consideration of this infinite system is related to earlier methods of solution — spherical harmonics, discrete-ordinates, etc. — which have been considered in the literature. (The latter methods are shown to be formally equivalent.) The practical advantage of defining the moments in the interval $0<\mu\leqq1$ and $0>\mu\geqq-1$ is emphasized. *S. Chandrasekhar.*

Piotrowski, S. The diffusion of light through a scattering layer of great optical thickness. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 303–306.

On a plane parallel atmosphere of optical thickness τ_1 and which scatters radiation in accordance with the conservative phase function $p=1+\omega_1\cos\Theta$, a parallel beam of radiation of net flux πF normal to itself is incident at an angle $\cos^{-1}\mu_0$. The net flux of diffuse radiation (i.e., radiation which has suffered one or more scattering processes) at an optical depth τ, is then given by

$$F(\tau)=\mu_0F\Big(e^{-\tau/\mu_0}+\frac{L_0}{3\mu_0}\Big),$$

where L_0 is a constant. By considering the solution in a finite approximation [cf. Chandrasekhar, Radiative transfer, Oxford, 1950, pp. 194–201; MR **13**, 136] and then passing to the limit of infinite approximation, the author shows that

$$L_0=-\sqrt{3}\frac{\mu_0H(\mu_0)}{(1-\frac{1}{3}\omega_1)\tau_1+2q(\infty)},$$

where $q(\infty)=0.7104\cdots$ and $H(\mu)$ is the H-function [cf. Chandrasekhar, loc. cit., Chap. V].

S. Chandrasekhar (Williams Bay, Wis.).

Ramakrishnan, Alladi. On stellar statistics. Astrophys. J. **122** (1955), 24–31.

In their theory of the fluctuations in brightness of the Milky Way, Chandrasekhar and Münch [same J. **112** (1950), 380–392; MR **12**, 644] formulated the following problem: Let stars and interstellar clouds occur with a uniform distribution. Let the system extend to a linear distance L in the direction of a line of sight. Let a cloud reduce the intensity of the light of the stars immediately behind it by a factor $q<1$. Let the occurrence of clouds with a transparency factor q be governed by a frequency function $\psi(q)$. Given all this, what is the probability distribution $g(I,L)$, of the observed brightness I. Assuming that stars occur so densely that they contribute a

uniform emission per unit volume, they derived the equation,

$$(1)\qquad g(u,\xi)+\frac{\partial g}{\partial u}+\frac{\partial g}{\partial \xi}=\int_0^1 \psi(q)g\left(\frac{u}{q},\xi\right)\frac{dq}{q},$$

where u is the brightness measured in suitable units and ξ is the average number of clouds in the direction of the line of sight. In this paper the author allows for the discrete nature of the sources contributing to the emission by introducing a function $\phi(\eta)$ such that $\mu\phi(\eta)d\eta$ is the probability that a given element of volume at ξ will contribute to an emission in the interval η, $\eta+d\eta$. Equation (1) is then generalized to

$$(2)\qquad \frac{\partial g}{\partial \xi}+(1+\mu)g(u,\xi)=\int_0^1 \psi(q)g\left(\frac{u}{q},\xi\right)\frac{dq}{q}+\mu\int_0^\infty g(u-\eta;\xi)\phi(\eta)d\eta.$$

By multiplying this equation by u^n and integrating, the author obtains the first three moments of g. In an appendix, S. K. Srinivasan obtains an explicit solution of the Laplacian transform of equation (2) in case $\psi(q)$ is a δ-function.

The paper also considers the modification of the analysis in case the picture of discrete absorbing clouds is replaced by a fluctuating density distribution [Chandrasekhar and Münch, ibid. **115** (1952), 103–123; MR **13**, 786]. If the optical thickness, τ_0, of a "single fluctuation" is much less than one, then the results derived on the picture of absorbing discrete clouds can be taken over for the fluctuating field by defining an equivalent optical thickness, $2\alpha^2\tau_0$, for a cloud, where α^2 is the mean square fluctuation in the density [cf. Ramakrishnan, ibid. **119** (1954), 443–455]. *S. Chandrasekhar.*

See also: Chandrasekhar and Prendergast, p. 561.

RELATIVITY

★ **Pantaleo, Mario. Introduzione generale.** Cinquant'anni di Relatività, 1905–1955, pp. XXI–L. Editrice Universitaria, Firenze, 1955.

★ **Severi, Francesco. Aspetti matematici dei legami tra relatività e senso comune.** Cinquant'anni di Relatività, 1905–1955, pp. 309–333. Editrice Universitaria, Firenze, 1955.

★ **Armellini, Giuseppe. La teoria della relatività nell'astronomia moderna.** Cinquant'anni di Relatività, 1905–1955, pp. 335–375. Editrice Universitaria, Firenze, 1955.

★ **Caldirola, Piero. Applicazioni e verifiche sperimentali della relatività ristretta.** Cinquant'anni di Relatività, 1905–1955, pp. 377–455. Editrice Universitaria, Firenze, 1955.

Finzi, Bruno. Cinquantenario della relatività di Einstein, 1905-1955. Confer. Sem. Mat. Univ. Bari nos. **11-12-13-14-15** (1955), 52 pp.

Five expository lectures, without derivations: 1. Tensor calculus in Euclidean spaces; 2. Tensor calculus in general spaces; 3. Einstein's gravitational theory; 4. The electromagnetic field in space-time; 5. Unified theories (Weyl, Kaluza, Einstein). *F. A. E. Pirani* (London).

Tonnelat, Marie-Antoinette. Sur les équations approchées de la théorie d'Einstein-Schrödinger. II. C. R. Acad. Sci. Paris **241** (1955), 1110–1112.

[For part I see same C.R. **241** (1955), 168–170; MR **17**, 201.] The author gives explicit expressions for the first and second order approximations to the Einstein generalized gravitational field equations. It is assumed that the electromagnetic field, described by the anti-symmetric part of the general tensor $g_{\mu\nu}$ has an expansion in terms of positive powers of a small parameter, ε, beginning with the first power. It is further assumed that the gravitational field has a corresponding expansion beginning with the zeroth power. *A. H. Taub* (Urbana, Ill.).

Bose, S. N. Solution d'une équation tensorielle intervenant dans la théorie du champ unitaire. Bull. Soc. Math. France **83** (1955), 81–88.

In the solution of the equations which determine the affine connection of Einstein's generalized theory of gravitation of the derivatives of the fundamental non-symmetric tensor $g_{\mu\nu}$ there occurs the equation

$$U_{\alpha\mu\nu}=T_{\alpha\mu\nu}+T_{\alpha\beta\lambda}C_\mu{}^\beta C_\nu{}^\lambda+C_\alpha{}^\lambda(T_{\lambda\beta\nu}C_\mu{}^\beta+T_{\lambda\mu\beta}C_\nu{}^\beta),$$

where $C_\lambda{}^\beta$ is a tensor determined by the anti-symmetric part of $g_{\mu\nu}$. The author discusses the solution of this equation for $T_{\alpha\mu\nu}$ as a linear function of $U_{\alpha\mu\nu}$.

A. H. Taub (Urbana, Ill.).

Horváth, J. I., und Moór, A. Entwicklung einer Feldtheorie begründet auf einen allgemeinen metrischen Linienelementraum. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 421–429, 581–587.

The authors endeavour to formulate a differential-geometric background for the bilocal field theories of Yukawa, in which the field variables depend on the coordinates $x_{(1)}{}^\mu$, $x_{(2)}{}^\mu$ of a pair of points in the 4-dimensional space-time continuum. New variables are introduced by putting $x^\mu=\frac{1}{2}(x_{(1)}^\mu+x_{(2)}^\mu)$, $r^\mu=x_{(1)}^\mu-x_{(2)}^\mu$. The manifold of elements (x, r) is called a Yukawa space Y, in which the (x, r) are considered as line-elements. In view of certain homogeneity considerations Y can be replaced by an equivalent space L of line-elements (x, v) where $v^\mu=\varrho r^\mu$ (ϱ an arbitrary positive constant). A sequence $(x_{(1)}(t), x_{(2)}(t))$ defines a sequence of line-elements $(x(t), v(t))$, which is interpreted as a curve in L with respect to the vector field $v(t)$. The length s of such a curve is defined after the introduction of a metric tensor $g_{\mu\nu}(x, v)$:

$$s=\int_{t_0}^{t_1}[g_{\mu\nu}(x,v)\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}dt\left(\dot{x}^\mu=\frac{dx^\mu}{dt}\right).$$

A covariant derivative depending on the $g_{\mu\nu}$ is defined, and necessary and sufficient conditions that L be a Finsler space are established. The autoparallel curves do not, in general, coincide with the geodesics of L. The covariant derivative leads in the usual manner to a curvature tensor. Let $\psi(x, v)$ characterise the field. A Lagrangian scalar density is defined by putting $\mathfrak{L}=L\sqrt{-g}$ (g=det $g_{\mu\nu}$), involving ψ and its derivatives. The usual variational principle is applied, assuming in addition that an absolute parallel displacement exists in L. In this manner the so-called field equations are to be derived. *H. Rund* (Toronto, Ont.).

Takeno, Hyôitirô. On groups of conformal transformations in spherically symmetric space-times. Tensor (N.S.) **5** (1955), 23–38.

This paper follows eleven others by the author on various properties of spherically symmetric space-times, and is concerned with those coordinate transformations of such space-times which leave them conformally invariant. The transformations are found explicitly, and space-times are classified according to the conformal transformation they admit. *A. G. Walker* (Liverpool).

Bertotti, B. On the motion of charged particles in general relativity. Nuovo Cimento (10) **2** (1955), 231–240.

This paper deals: a) with the procedure of obtaining the self-reaction terms for the motion of a charged particle the method is simpler than that used by Infeld and (Wallace [Phys. Rev. (2) **57** (1940), 797–806; MR **1**, 274]); b) with the problem of motion of two charged particles with the aid of the "geodesic method" if the gravitational forces, the electromagnetic forces and their mixture is taken into account. *L. Infeld* (Warsaw).

Goldberg, Joshua N. Gravitational radiation. Phys. Rev. (2) **99** (1955), 1873–1883.

The author investigates the existence of gravitational radiation within the framework of the Einstein-Infeld-Hoffmann method: Following a prescription introduced by Infeld, the radiation terms are begun with a harmonic function of the time alone. He finds that these terms do not have physical significance if they are introduced in the scalar of longitudinal components of the gravitational potentials. However, if the radiation terms are introduced in the fifth order of the transverse-transverse components, one finds apparently a contribution to the metric tensor of the seventh order, that is ${}_7\gamma_{rs}$ in (45a). However, the author does not seem to notice that this ${}_7\gamma_{rs}$ is a harmonic function which can be arbitrarily added or subtracted at each higher-approximation step. Therefore the author's argument seems illusory. *L. Infeld* (Warsaw).

Scheidegger, Adrian E. Gravitational radiation. Phys. Rev. (2) **99** (1955), 1883–1885.

The contents of this paper is a polemic with the paper reviewed above in which, however, according to the reviewer the important point that the function ${}_7\gamma_{rs}$ (which is different from zero) is a harmonic function is not mentioned. *L. Infeld* (Warsaw).

Just, Kurt. Hertzscher Dipol im expandierenden Raum. Z. Physik **142** (1955), 266–276.

The author discusses particular solutions of the Maxwell field equations corresponding to a Hertzian dipole in a cosmological space with a varying gravitational "constant". The influence of the Maxwell field on the structure of space-time is ignored. Detailed discussions of the behavior of the solutions in a variety of cosmological models are given. *A. H. Taub* (Urbana, Ill.).

Heckmann, O., und Schücking, E. Bemerkungen zur Newtonschen Kosmologie. I. Z. Astrophys. **38** (1955), 95–109.

The authors generalize the cosmological theory of Milne and McCrea [Milne, Quart. J. Math. Oxford Ser. **5** (1934), 64–72; McCrea and Milne, ibid. **5** (1934), 73–80], which ostensibly treats a spatially uniform and isotropic distribution of matter on the basis of Newtonian theory, by dropping the assumption of isotropy. In this way they are able to derive a Newtonian model, analogous to Gödel's relativistic model [Rev. Mod. Phys. **21** (1949), 447–450; MR **11**, 216], in which the universe as a whole rotates about an axis fixed in space, as well as a more general model, for which no relativistic analogue yet exists, in which the rotation is superposed on a general expansion.

Layzer [Astr. J. **59** (1954), 268–270; MR **16**, 410] has pointed out that Newtonian mechanics cannot be applied in a self-consistent way to a uniform, unbounded distribution of matter, and McCrea [Nature **175** (1955), 466; Astr. J. **60** (1955), 271–274; MR **17**, 421] subsequently modified his theory by postulating a model universe in which matter is confined to the interior of a sphere of finite, though arbitrarily large, radius. The authors assert that their treatment of the problem vindicates the application of Newtonian mechanics to a uniform, unbounded distribution of matter, and renders McCrea's restriction of the theory to a finite system unnecessary, but they present no argument in support of this assertion. *D. Layzer* (Cambridge, Mass.).

McCrea, W. H. On newtonian frames of reference. Math. Gaz. **39** (1955), 287–291.

Reinterpreting earlier work of Milne and McCrea [Milne, Quart. J. Math. Oxford Ser. **5** (1934), 64–72; McCrea and Milne, ibid. **5** (1934), 73–80], the reviewer has shown that the dynamical properties of a uniform, pressure-free, spherical system, consisting of gravitating particles, are the same when the system is embedded in empty space as when it forms part of an unbounded distribution of matter, uniformly filling all space; and that these properties are given, without approximation, by Newtonian theory [Astr. J. **59** (1954), 268–270; MR **16**, 410]. An obvious corollary of this statement is the theorem that any spherical subregion of such a system behaves as if it were embedded in empty space. The author gives a detailed derivation of this theorem, using only Newtonian mechanics and kinematics, and discusses its significance. He points out that while there exists only one family of strictly Newtonian frames of reference, any frame moving with a particle of the system 'serves as a newtonian frame for the description of mechanical phenomena throughout a region that may be taken to be arbitrarily large,' even though such a frame is in general accelerated with respect to the true Newtonian frames. *D. Layzer* (Cambridge, Mass.).

Rayner, C. B. The effects of rotation of the central body on its planetary orbits, after the Whitehead theory of gravitation. Proc. Roy. Soc. London. Ser. A. **232** (1955), 135–148.

Synge [same Proc. **211** (1952), 303–319; MR **14**, 99] has solved the problem of the gravitational field due to a finite, static, spherically symmetric distribution of matter, according to Whitehead's theory. The author calculates the more difficult problem of a uniformly rotating sphere. Then the advance of the perihelion of an equatorial orbit is calculated. The result is essentially the same as that obtained by Lense and Thirring [Phys. Z. **19** (1918), 156–163] by the use of Einstein's linearized law of gravitation. *L. Infeld* (Warsaw).

See also: Polvani, p. 445; Straneo, p. 445; Einstein, p. 446; Aliotta, p. 448; Gürsey, p. 558; Carini, p. 561.

MECHANICS

Mihăilă, N. Observations sur l'aspect cinématique des mécanismes. Gaz. Mat. Fiz. Ser. A. **7** (1955), 584–590. (Romanian. Russian and French summaries)

Sălăgean, Ioan, and Sălăgean, Traian. On the equilibrium of a rigid solid body resting on a plane surface. Gaz. Mat. Fiz. Ser. A. **7** (1955), 541–552. (Romanian. Russian and French summaries)

Kurth, R. Verallgemeinerung eines elementaren Satzes von Laplace. Elem. Math. **10** (1955), 127–128.

Let m_1, m_2, and m_3 be three non-collinear mass particles which act upon each other in such a way that the force exerted by m_i on m_j is directed along the line joining these two particles and is equal in magnitude but opposite in sense to the force exerted by m_j on m_i $(i, j=1, 2, 3; i\neq j)$. Also let k_i denote the resultant force exerted on m_i by the other two particles. The author's elementary considerations then show that k_1, k_2, and k_3 are concurrent. If the force exerted by m_i on m_j is in magnitude directly proportional to the product of their masses and if in addition its ratio to the distance between m_i and m_j is a strictly monotonic function of this distance, then the point of concurrency of k_1, k_2, and k_3 coincides with the mass center, if and only if the particles form an equilateral triangle. The special case where the forces are those of Newtonian gravitation was treated by Laplace.

D. C. Lewis (Pacific Palisades, Calif.).

Pihl, Mogens. Classical mechanics in a geometrical description. Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), no. 12, 26 pp. (Danish. English summary)

The possibilities of giving a simple geometrical description of classical mechanics are discussed, starting from a historical point of view. A geometrization of this kind requires the general so-called Finsler-geometry [Thesis, Göttingen, 1918; cf. MR 13, 74], a short account of whose main features, adapted to this purpose, is given in the paper. It is shown that the introduction of cyclic coordinates allows the reduction to the more specialized Riemannian geometry. The author is here concerned with a slight change of an idea, originally formulated by J. J. Thomson [Applications of dynamics to physics and chemistry, Macmillan, London, 1888], which in particular forms the basis of H. Hertz's endeavours to eliminate the concept of force in mechanics [H. Hertz, Die Prinzipien der Mechanik, Gesammelte Werke, Vol. III, 2.ed., Leipzig, 1912]. It is finally shown that this geometrization also can be applied to time-dependent, holonomic dynamical systems, leading to a geometrical description which already has been studied by L. P. Eisenhart [Ann. of Math. (2) **30** (1929), 591–606]. *R. Gran Olsson* (Trondheim).

★**Kondo, Kazuo. On the theoretical investigation based on abstract geometry of dynamical systems appearing in engineering.** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 425–432. Science Council of Japan, Tokyo, 1954.

This paper presents various ideas on the relations between the geometries of non-Riemannian and Finsler spaces and the dynamical systems of engineering. First, the author shows that the second-order differential equations of a dynamical system with linear resistance can be realized by the equations of paths in an $(n+1)$-dimensional affine space. Secondly, the Jacobi conditions for the infinitesimal displacement of a geodesic (as formulated by T. Levi-Civita) in an $(n+1)$-dimensional affine space are analyzed as the equations of hunting of a dynamical system. As noted by the author, similar results were given by G. Kron [J. Math. Phys. **13** (1934), 103–194]. The main problem of the author is to determine spaces which can be used to represent engineering systems possessing non-linear resistance. By proper modification of the Hamiltonian of a mechanical system and use of a contact transformation, the geodesics of an $(n+1)$-dimensional Riemann space are transformed into the geodesics of an $(n+1)$-dimensional Finsler space. The latter equations are shown to be similar to the equations of an engineering system (second-order equations) with non-linear resistance. Further, it is shown how one may introduce a potential energy function into the dynamical equations by use of a proper metric tensor in Finsler space. Also the author sketches a scheme for introducing a Riemannian geometry into the treatment of statistical systems. Possible expressions for the potential, kinetic and heat energy are introduced. *N. Coburn.*

★**Fujinaka, Megumi. On Finsler spaces and dynamics with special reference to equations of hunting.** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 433–436. Science Council of Japan, Tokyo, 1954.

In this paper, the author discusses: (1) the equations of geodesics in Finsler space; (2) the equations of hunting in Finsler space. First, the equations of the geodesics of Finsler space [see the paper reviewed above] are analyzed in some detail and analogies with geodesics in Riemann space are shown. Then, the equations of hunting in Finsler space are shown to be extensions involving non-linear resistance of the hunting equations used by G. Kron [J. Math. Phys. **13** (1934), 103–194]. *N. Coburn.*

Magnus, K. Beiträge zur Dynamik des kräftefreien, kardanisch gelagerten Kreisels. Z. Angew. Math. Mech. **35** (1955), 23–34. (English, French and Russian summaries)

The paper deals with the influence of the gimbals G_2 (outer) and G_1 (inner) on the motion of a centered gyroscope G_0 under no forces except for the reactions of the smooth bearings of the gimbals. The following account is tentative because of a number of misprints, unidentified symbols, and several statements not intelligible to the reviewer. It is not stated whether the gimbals are axially symmetric. Let ϕ_2 be the angular displacement of G_2 relative to the inertial reference frame, ϕ_1 the angle of the axis z_0 of G_0 with the plane through the center 0 perpendicular to the axis z_2 of G_2. Let z_1 be the common axis of G_1 and G_2. For $\phi_1=0$, the three axes are perpendicular. The first problem treated is that of a symmetric gyro. The equations of motion are written down, and integrated by elliptic quadratures. Under the assumption that ϕ_1 differs little from a constant b (small nutation), an approximate solution by successive iterations is given, the error being of the second order in the amplitude a of ϕ_1. It is found that

$$\phi_1=b+a\cos nt,\ \phi_2=(I_1/I_0)^{\frac{1}{2}}a\sin nt-a^2Jt\sin b\cdot(I_{z2}+I_{z1})/2I_0^2,$$

where $n=J\cos b/(I_1I_0)^{\frac{1}{2}}$, J is the constant angular momentum of the gyro about z_0, $I_0=I_z\cos^2 b+(I_{z2}+I_{z1})\cdot$

$\sin^2 b$. Here I_s is the moment of inertia of the whole system about z_2 in the position $\phi_1=0$, I_{s2} the moment of inertia of G_2 about z_2, I_{s1} the moment of inertia of G_1 about z_0, and I_1 the moment of inertia of the system (G_0, G_1) about the common axis of G_1 and G_2. It is seen that G_2 oscillates about a plane uniformly rotating about z_2.

The second problem is that of the stability of a non-symmetric gyroscope whose angular velocity is nearly axial. The second-order terms in the equatorial components of the angular velocity are neglected, and a system of two linear equations for these components is obtained with coefficients linear in $\cos 2\phi_0$ and $\sin 2\phi_0$, ϕ_0 being the angular displacement of the gyro about its axis. The Poincaré-Lyapunov theory of perturbations is then applied (the parameter here is a ratio of linear combinations of moments of inertia). The results are too involved to be reproduced here. *A. W. Wundheiler.*

Patni, G. C. On the solution of the system of equations in internal ballistics. Proc. Nat. Inst. Sci. India. Part A. **21** (1955), 196–217.

See also: Letov, p. 487; Łojasiewicz, p. 485; Suter, p. 521; Roth-Desmeules, p. 541; Sklyanskiĭ, p. 542.

Fluid Mechanics, Acoustics

Ertel, Hans. Kanonischer Algorithmus hydrodynamischer Wirbelgleichungen. S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1954**, no. 4, 11 pp. (1955).

The reviewer's generalization [Z. Angew. Math. Phys. 2 (1951), 109–114; MR **12**, 761] of the author's first vorticity theorem is rediscovered and shown to include a number of known vorticity theorems as special cases [cf. the reviewer, op. cit. and also § 79 and § 98 of The kinematics of vorticity, Indiana Univ. Press, 1954]. *C. A. Truesdell* (Bloomington, Ind.).

Gubin, V. I. A generalization of Bjerknes' theorem. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 12 (1953), 84–90. (Russian)

L'auteur essaie de généraliser le célèbre théorème de Bjerknes des champs solénoïdaux en introduisant le frottement. En admettant quelques restrictions l'auteur arrive à la conclusion que la présence des forces de frottement n'a aucune influence sur le nombre des solénoïdes à l'intérieur d'un contour fermé. *M. Kiveliovitch.*

Suharevs'kiĭ, I. V. On a boundary problem of hydrodynamics. I. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 416–418. (Ukrainian. Russian summary)

The author investigates the analytic nature of the solutions of the boundary problem of finding the velocity of the unseparated plane flow of an ideal incompressible fluid about a simple piecewise-smooth contour C $(z=z(s))$ with an angular point $z_0=z(0)=z(l)$ when there are a discrete or continuous distribution of singularities in the region of the flow. It is established that if $\arg dz/ds$ and the boundary values of the complex velocity of the undisturbed flow satisfy for $0\leq s\leq l$ a Lipschitz condition with some positive exponent sufficiently close to one, then there exists a unique flow with complex velocity continuous in the closed region of the flow and equal to zero at the angular point.

If all singularities of the flow occur at a positive distance from the contour, then the boundary value of the complex velocity satisfies a Lipschitz condition with a positive exponent. From this one may conclude solvability in the class of continuous functions of the integral equations

$$u(s)-\frac{1}{\pi}\int_C u(\sigma)\frac{\partial \ln|\xi-z|}{\partial n_s}d\sigma=2\,\mathrm{Re}\,[w_0(z)z'(s)],$$

$$-\frac{1}{\pi}\int_C u(\sigma)\frac{\partial \ln|\xi-z|}{\partial s}d\sigma=2\,\mathrm{Im}\,[w_0(z)z'(s)],$$

where $u(s)=w(z)z'(s)$, and n is the exterior normal vector; here $w(z)$ is the complex velocity of the motion and $w_0(z)$ is the complex velocity of the motion without the contour. *Yu. D. Sokolov* (RŽMeh **1955**, no. 3006).

Patraulea, N. Théorie des mouvements aux sillages presque rectilignes. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 139–148. (Romanian. Russian and French summaries)

The author deals with the problem of slightly perturbed movements of incompressible perfect fluids in the presence of several permeable, or opaque bodies. It is shown that, in certain cases, the velocity potential of the irrotational motion can be continued analytically to the exterior of the bodies. The expression for the pressure is determined by means of the Bernoulli formula, which is supposed to be valid for large perturbations, too. An application is made to the case of the flow around a helix. *K. Bhagwandin* (Oslo).

Carafoli, E., et Patraulea, N. L'équation de la circulation autour d'une aile à fuselage central. Com. Acad. R. P. Române **2** (1952), 249–255. (Romanian. Russian and French summaries)

The author assumes that the fuselage may be replaced by an infinite cylinder normal to the wing and parallel to the flow at infinity. Far behind the wing the perturbation of uniform flow is essentially plane flow about the fuselage induced by the trailing vortex system. By means of a conformal mapping in planes normal to the fuselage which carries a cross-section of the fuselage onto a slit, the author reduces the problem to that of determining the spanwise distribution of circulation about an image wing without a fuselage but with appropriately modified chord and twist. When this has been found by standard methods, the induced drag and the spanwise lift distribution can easily be computed for the original wing-fuselage combination. *J. Giese* (Aberdeen, Md.).

Carafoli, E., et Patraulea, N. Le théorème de la résistance minimum des systèmes portants complexes. Com. Acad. R. P. Române **2** (1952), 441–446. (Romanian. Russian and French summaries)

The authors wish to minimize the induced drag of a system of lifting lines with prescribed total lift L in a given direction D and prescribed rolling moment M about a given point O. For a system of wings the local component of induced velocity normal to the wings must arise from a superposition of translation at some velocity w parallel to D and rotation with some angular velcoity ω about O. For a system composed of wings and infinite cylindrical fuselages they propose to solve their problem by superposing two flows obtained as follows. 1) Treat the fuselages as lifting surfaces and find the optimum flow with appropriate w and ω for the entire wing fuselage system.

2) For the fuselage system alone find the flow with induced velocity corresponding to $-w$ and $-\omega$. The reviewer is not convinced by the authors' argument that this superposition will minimize the induced drag. *J. Giese.*

Muggia, Aldo. Remark on the theory of lifting surfaces. NACA Tech. Memo. no. **1386** (1956), 11 pp.
Translation of Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **87** (1953), 193–199; MR **15**, 998.

★ **Weissinger, J. Zur Aerodynamik des Ringflügels. Die Druckverteilung dünner, fast drehsymmetrischer Flügel in Unterschallströmung.** Forschungsberichte des Wirtschafts- und Verkehrsministeriums Nordrhein-Westfalen, Nr. 198. Westdeutscher Verlag, Köln und Opladen, 1955. 30 pp. DM 9.00.
A lifting-surface analysis of the ring airfoil is made on the basis of an equivalent vortex sheet. Fourier-series expansion in the angular coordinate reduces the resulting, two-dimensional integral equation to a set of uncoupled integral equations in the axial coordinate (such a separation of variables is not possible for a finite, plane lifting surface without further approximation). It is shown that the formulation reduces essentially to lifting-line theory as the chord/radius ratio tends to zero [in which approximation the problem was solved by H. Ribner, J. Aero. Sci. **14** (1947), 529–530; MR **9**, 115] and to slender-wing theory as this ratio tends to infinity. Numerical solutions to the integral equations are tabulated for constant angle of attack [the non-lifting problem having been treated previously by D. Küchemann and J. Weber, Aerodynamics of propulsion, McGraw-Hill, New York, 1953]. Approximations to the section lift and moment based on the authors well known $\frac{3}{4}$ chord method are shown to compare quite favorably with the numerical results.
J. W. Miles (Los Angeles, Calif.).

Collins, W. D. On the steady rotation of a sphere in a viscous fluid. Mathematika **2** (1955), 42–47.
L'auteur établit une relation entre le mouvement lent et rectiligne d'un corp de révolution autour son axe dans un milieux visqueux, et le mouvement lent le long de son axe dans une fluide parfaite. En cas de rotation la pression est constante pendant que les composants de vélocité U et W sont zéro. Le composant $V\cdot\tilde{\omega}=\chi$ ($\tilde{\omega}$, ϕ et z sont les coordonnées cylindriques) est déterminé par l'équation

$$\left[\frac{\partial^2}{\partial\tilde{\omega}^2}-\frac{1}{\tilde{\omega}}\cdot\frac{\partial}{\partial\omega}+\frac{\partial^2}{\partial z^2}\right]\chi=0.$$

L'auteur déduit la même équation pour la fonction de courant en cas de mouvement rectiligne.
Enfin, la rotation d'un sphère dans un milieux visqueux est traité. Les composantes u, v et w sont développés en series de Fourier par rapport à R (=numéro Reynolds). Les expressions approximatives de troisième ordre sont obtenue. L'expression pour le couple est donné par

$$g=g_1+g_3R^2+g_5R^4+\cdots,$$

où $g_1=-25.1328$, $g_3=-2.0944\times10^{-2}$, $g_5=+1.8956\times10^{-5}$. *K. Bhagwandin* (Oslo).

Rott, Nicholas. Unsteady viscous flow in the vicinity of a stagnation point. Quart. Appl. Math. **13** (1956), 444–451.
In this note, the author considers the problem of two-dimensional steady viscous flow in the vicinity of a stagnation point at which the velocity in the main stream increases linearly with the distance. It is noticed by the author that, for the steady flow at the edge of the boundary layer, if the plate performs harmonic or uniform motion parallel to itself, a solution for the unsteady problem can be found by superposition of two functions such that the full Navier-Stokes equations are satisfied. In the oscillating case, the shear at the wall is calculated at both low and high frequencies. It is noted that, in spite of the main stream, the character of the flow at very high frequencies is essentially that of the Stokes solution.
Y. H. Kuo (Ithaca, N.Y.).

Dolidze, D. E. Remark on the papers of D. E. Dolidze, "A nonlinear boundary problem for unsteady motion of a viscous incompressible fluid," Prikl. Mat. Meh., v. 12, no. 2, 1948, and "Unsteady motion of a viscous fluid created by a rotating disc", Prikl. Mat. Meh., v. 18, no. 3, 1954. Prikl. Mat. Meh. **19** (1955), 764. (Russian)
Further remarks on two earlier papers [ibid. **12** (1948), 165–180; **18** (1954), 371–378; MR **9**, 630; **16**, 1061].

★ **Hasimoto, Hidenori. Rayleigh's problem for a plate of finite breadth.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 447–452. Science Council of Japan, Tokyo, 1952.

★ **Inui, Takeo, and Toba, Kaoru. The steady and slow motion of viscous fluid past a prolate spheroid.** Proceedings of the Second Japan National Congress for Applied Mechanics, 1952, pp. 233–238. Science Council of Japan, Tokyo, 1953.

Gheorghiță, Șt. I. Quelques mouvements en milieux poreux non homogènes. Acad. R. P. Romîne. Bul. Ști. Secț. Ști. Mat. Fiz. **6** (1954), 823–838. (Romanian. Russian and French summaries)
The author's investigation is based on solutions of the Laplace equation and the Darcy hypothesis. The conditions for the velocity potentials for a fluid in two media with different filtration coefficients are stated. The necessary conditions on the surface of separation of the media are also obtained [cf. O. D. Kellogg, Foundations of potential theory, Springer, Berlin, 1929].
Furthermore, the author discusses the particular case of a circular porous cylinder placed in a porous medium with different permeability-coefficients. The velocity potential at great distances from the cylinder is normal to the axis of the cylinder. In the case of several media (where the surfaces of separation are co-axial cylinders) the author shows that the velocity in the medium, containing the mutual axis of the cylinders, is parallel to the velocity at infinity. The similar problems for the case of an ellipse and a permeable plate are also treated. In the case of impermeability, the results are in agreement with those obtained by means of perfect-fluid theory. In the case of a permeable sphere placed in a porous medium, the author obtains the solution by means of Kelvin's transformation [cf. Kellogg, loc. cit.; A. N. Tihonov and A. A. Samarski, Equations of mathematical physics, 2nd ed., Gostehizdat, Moscow, 1953; MR **16**, 364]. Fourier-series expansions are employed throughout the paper. *K. Bhagwandin* (Oslo).

★**Kataoka, Masaharu. On the boundary layer of a plate placed parallel to the flow.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 453–456. Science Council of Japan, Tokyo, 1952.

In this paper the author attempts to find a more general solution for the viscous flow around a flat plate than the usual boundary-layer solution. Certain "generalized" equations of motion are presented, without derivation and with little explanation of some unorthodox features. On the basis of several arbitrary approximations and assumptions (some of doubtful validity in the reviewer's opinion, e.g., a tacit assumption of constant entropy), formal analytical results are derived. No discussion of the significance of these results is given. *D. W. Dunn.*

Schlichting, H. Laminar flow about a rotating body of revolution in an axial airstream. NACA Tech. Memo. no. **1415** (1956), 43 pp.

Translation of Ing.-Arch. **21** (1953), 227–244; MR **15**, 477.

Dem'yanov, Yu. A. Self-similar problems of an unsteady boundary layer of a compressible gas. Prikl. Mat. Meh. **19** (1955), 760–761. (Russian)

★**Tatsumi, Tomomasa. Theory of isotropic turbulence with the normal joint-probability distribution of velocity.** Proceedings of the Fourth Japan National Congress for Applied Mechanics, 1954, pp. 307–311. Science Council of Japan, Tokyo, 1955.

The author investigates the problem of isotropic turbulence under the assumption that fourth-order mean values are related to second-order ones in the same manner as for a normal joint-probability distribution. This statistical hypothesis renders determinate the system of equations for the two-point double correlations and three-point triple correlations. The author derives the corresponding equations for the Fourier transforms of these correlations, and after a lengthy analysis obtains an integral equation giving the energy spectrum function for all time, provided the initial values of this function and another scalar function associated with the third-order Fourier transform are specified. In the case of vanishing viscosity and for particular initial conditions, two terms of a power-series solution for small values of the time are derived. [This problem has also been investigated on the basis of the above statistical hypothesis by Proudman and Reid, Philos. Trans. Roy. Soc. London. Ser. A. **247** (1954), 163–189; MR **16**, 299.] *D. W. Dunn.*

Monin, A. S. On diffusion with finite velocity. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1955**, 234–248. (Russian)

L'auteur étudie la diffusion turbulente comme résultat d'action d'un certain nombre de petits courants qui diffèrent par la grandeur et la direction. Avant tout l'auteur étudie la présence de deux types de courants dans un milieu homogène. Ensuite, en vue d'une étude plus détaillée, il étudie n types de courants en supposant que l'ordonnée de la particule ainsi que le numéro du type de courant forment un processus de Markoff à deux dimensions. Comme application l'auteur examine la diffusion dans une couche d'air au voisinage du sol. Il est intéressant de constater que pour deux courants qui ne diffèrent que par leur direction les équations se ramènent à une solution de l'équation des télégraphistes.

M. Kiveliovitch (Paris).

Todeschini, Bartolomeo. Le condizioni di isentropicità e di irrotazionalità nell'aerodinamica piana. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 619–630.

This paper contains a discussion of the equations governing two-dimensional steady gas flow where viscosity and heat conduction are absent. The material covered is familiar, but the author does add a new proof of the result of Gilbarg that, locally, a non-isentropic irrotational flow is either uniform or circulatory [Amer. J. Math. **71** (1949), 687–700; MR **11**, 61]. *J. B. Serrin.*

Bagdoev, A. G. Penetration of a narrow cone into a compressible liquid. Vestnik Moskov. Univ. **10** (1955), no. 10, 65–69. (Russian)

"Vertical penetration of a narrow cone into a compressible fluid with free surface is considered. The resulting motion of the fluid is supposed to be small and potential".

Introduction.

Bergman, Stefan. On representation of stream functions of subsonic and supersonic flows of compressible fluids. J. Rational Mech. Anal. **4** (1955), 883–905.

Il s'agit d'un écoulement plan d'un fluide parfait compressible étudié par la méthode de l'hodographe. La fonction de courant ψ vérifie une équation du second ordre linéaire. Dans la première partie concernant des écoulements subsoniques, les variables sont choisies de manière à mettre sous la forme d'un laplacien l'ensemble des termes du second degré: ψ est défini dans le plan pseudo-logarithmique, θ désigne l'angle de la vitesse avec une direction fixe, la variable λ est lié au module de la vitesse; un changement de fonction permet de mettre l'équation sous la forme $\Delta\psi^*+4F(\lambda)\psi^*=0$. L'auteur considère alors le cas où le domaine de l'hodographe est limité par des segments parallèles aux axes, ce qui implique, dans le plan physique, un écoulement limité par des parois rectilignes et des lignes d'égale vitesse. Il fournit de ce problème une solution théorique valable même si la fonction de courant est multiforme, ce qui conduit à étudier les solutions d'une équation aux dérivées partielles du second ordre du type elliptique définies sur une surface de Riemann. Dans la seconde partie, l'auteur considère des problèmes relatifs aux écoulements transsoniques et supersoniques et doit alors remplacer la variable λ par une autre mieux adaptée à ces problèmes. L'auteur améliore un résultat du à L. Bers [NACA Tech. Note no. **2058** (1950); MR **12**, 61] sur les bornes de la solution d'un problème de Cauchy en fonction des bornes des données de Cauchy, lorsque les coefficients de l'équation vérifient certaines inégalités assez générales.

P. Germain (Paris).

Legendre, Robert. Calcul d'une grille d'aubes pour une distribution de nombres de Mach subsoniques donnée en fonction du potentiel. Rech. Aéro. no. **47**, 3–9 (1955).

Le calcul d'une grille d'aubes, dont le profil d'aile d'avion est un cas particulier, est largement simplifié si la distribution des nombres de Mach est donnée en fonction du potentiel plutôt qu'en fonction de l'abscisse. Deux intégrations successives fournissent le champ des vitesses, puis le champ de l'écoulement. (Author's summary.)

L. M. Milne-Thomson (Greenwich).

Iacob, Caius. Sur les écoulements subsoniques, à circulation, des fluides compressibles. Com. Acad. R. P. Române 1 (1951), 741–746. (Romanian. Russian and French summaries)

Let $f=\phi+i\omega$ where ϕ and ψ are the velocity potential and stream function for plane flow at Mach number $M_0=U/c_0$ about a profile C_0. Let $z=x+iy$ and $\omega=\theta+i\log V$, where $Ve^{i\theta}$ is the velocity. Expand $\omega=\sum_0^\infty \omega_n M_0^{2n}$, where $\omega_n=\omega_n(f,\bar{f})$. The equations of irrotational flow yield a sequence of partial differential equations which imply $\omega_0=\omega_0(f)$, while ω_1 involves an arbitrary analytic function $g(f)$, etc. The author uses the following correspondence with an incompressible flow to determine ω_0, g, etc. Let $f(\zeta)$ be the complex velocity potential for incompressible circulatory flow at velocity U_0 about C_0 in a ζ-plane. Let $df/d\zeta=h(\zeta)e^{-i\omega_0(f)}$, where $h(\zeta)\neq 0$ is holomorphic outside C_0. Then

$$dz=h(\zeta)d\zeta+M_0^2[ig(\zeta)h(\zeta)d\zeta-0.25U^{-2}\overline{(df/d\zeta)}^2\bar{h}(\zeta)^{-1}d\bar{\zeta}]+\cdots.$$

One choice of U_0, $g(\zeta)$, and $h(\zeta)$ yields a correspondence previously developed by J. Leray [J. Math. Pures Appl. (9) **28** (1949), 181–191; MR **11**, 475]. *J. Giese.*

Woods, L. C. Compressible subsonic flow in two-dimensional channels. II. The application of the theory to problems of channel flow. Aero. Quart. **6** (1955), 254–276.

Cet article contient les applications de la théorie développée par l'auteur dans la première partie, [Aero. Quart. **6** (1955), 205–220; MR **17**, 206]. Rappelons que le fluide considéré est le fluide fictif de Chaplygin-Karman-Tsien. Il est assez facile d'obtenir la solution du problème lorsque la fonction donnant les variations de l'angle des parois avec une direction fixe est constante par morceaux. On obtient également un résultat relativement simple si sur certains arcs de la paroi on remplace la condition précédente par la suivante: la pression est une fonction constante par morceaux. Dans le cas d'une paroi courbe donnée, il faut résoudre numériquement une équation intégrale assez compliquée. L'auteur propose une méthode d'itération, qu'il applique d'ailleurs au cas délicat de l'écoulement autour d'un cylindre placé dans un canal limité soit par deux parois parallèles et rectilignes, soit par deux lignes de jet. Cette étude conduit à des résultats intéressants du point de vue pratique. On constate par exemple que la correction de "blocage" estimée habituellement à partir de la théorie linéaire fournit encore un résultat convenable dans le second cas, mais ne convient plus dans le premier. Enfin l'auteur propose une méthode pratique pour déterminer de façon satisfaisant la forme des parois qu'il faut obtenir pour éviter les corrections dans les expériences faites dans des souffleries à parois variables. *P. Germain* (Paris).

Woods, L. C. On the theory of two-dimensional wind tunnels with porous walls. Proc. Roy. Soc. London. Ser. A. **233** (1955), 74–90.

L'auteur, poursuivant ses études sur les écoulements dans les canaux [confer la précedente analyse], envisage le cas où une portion de chacune des parois est poreuse. Le but pratique de cette étude est de déterminer dans quelles conditions on peut supprimer les corrections de blocage qui deviennent importantes lorsque l'écoulement dans une soufflerie atteint des vitesses voisines de celle du son. Un profil symmétrique est placé à égale distance de deux parois parallèles; le long de la partie poreuse des parois, on suppose que la composante normale de la vitesse est proportionnelle à la différence de pressions existant de part et d'autre de la paroi; on suppose également qu'un certain débit fluide est extrait du canal, ceci afin de compenser éventuellement les effets de blocage dus à la présence du sillage en aval du profil. Après avoir linéarisé le problème, on est conduit à trouver une fonction de variable complexe définie dans une bande, connaissant les valeurs prises au contour par une combinaison linéaire et à coefficients constants de la partie réelle et de la partie imaginaire. Ce problème se ramène, comme on sait, à un problème de Dirichlet, dont l'auteur a déjà donné une solution dans ses précédents articles. Des applications de cette théorie générale permettent de trouver à quelles conditions doivent satisfaire les paramètres pour éliminer la correction due au blocage. *P. Germain* (Paris).

Frankl', F. I. An example of a transsonic gas flow with the region of supersonic velocities bounded downstream by a density discontinuity terminating within the flow. Prikl. Mat. Meh. **19** (1955), 385–392. (Russian)

Let φ and ψ be velocity potential and stream function, θ the inclination of the velocity, and η distorted difference from critical sonic speed. The author constructs an approximation to a flow which for $\psi>0$ is completely subsonic, but for $\psi<0$ contains a straight normal shock. For the approximate system (*) $\partial\varphi/\partial\theta=-C\partial\psi/\partial\eta$, $\partial\varphi/\partial\eta=C\eta\partial\psi/\partial\theta$ he seeks a solution (**) $\psi(\eta,\theta)=\varrho^\nu f_\nu(\theta/\varrho)$ adjacent to the positive η-axis, (***) $\psi=K(-\varrho^2)^{\frac{1}{2}\nu}f_\nu(\theta/\varrho)$ adjacent to the negative η-axis, where $\varrho^2=\theta^2+4\eta^3/9$, and C and K are constants. Assume $\varphi=0$ and $\theta=0$ on the shock, so $\partial\psi/\partial\theta=0$ there also. Then $f_\nu=F(\frac{1}{4}\nu+\frac{1}{8};\ -\frac{1}{4}\nu;\ \frac{1}{2};\ \theta^2/\varrho^2)$ and the entire flow lies in $\theta\geqq 0$. Since the sub- and supersonic parts of the streamline $\psi=0$ must map onto two curves $\theta^2/\varrho^2=$constant in the first quadrant of the θ plane, then $8/3\leqq\nu\leqq 14/3$. To the order of other approximations the velocity relation at the shock implies $\psi(\eta,0)=\psi(-\eta,0)$. Since the solutions (**) and (***) must also match on the characteristic of the system (*) through the origin, this implies $\nu=8/3$ or 4. The first value yields no transition through the speed of sound. For $\nu=4$, $\psi=-\eta^4+6\eta\theta^2$, and the corresponding velocity distribution is triple valued in part of the supersonic region of the physical plane. By excising from each streamline $\psi<0$ the loop at its point of self-intersection, the author produces a single-valued flow field with a second line of discontinuities, said to be negligible to the order of other approximations. *J. Giese* (Aberdeen, Md.).

Hida, Kinzō. A derivation of Guderley-Barish's solution for the sonic flow on the basis of the thin-wing expansion method. J. Phys. Soc. Japan **10** (1955), 1011–1017.

G. Guderley et D. T. Barish ont donné les premiers termes du développement asymptotique du potentiel des vitesses d'un écoulement transsonique, plan ou de révolution, valable loin de l'obstacle à l'infini amont [J. Aero. Sci. **20** (1953), 491–499]. Ce résultat était obtenu à partir de l'approximation usuelle dans l'étude des écoulements transsoniques. L'auteur du présent article retrouve ce même résultat par une autre voie. Utilisant la méthode de I. Imai [J. Phys. Soc. Japan **8** (1953), 537–544; MR **15**, 479] pour étudier par approximations successives les écoulements subsoniques, il étudie le comportement à l'infini amont des différents termes. L'auteur indique que certains coefficients numériques du développement proposé par G. Guderley et D. T. Barish pour les écoulements de révolution sont erronnés. *P. Germain* (Paris).

Ray, M. Flow of a compressible fluid around a corner. Proc. Nat. Inst. Sci. India. Part A. 21 (1955), 155–160.

L'auteur étudie le problème de l'écoulement plan supersonique autour d'un angle, dans le cas d'un fluide visqueux et conducteur de chaleur; il construit une solution dans laquelle les fonctions inconnues ne dépendent pas de la distance au sommet de l'angle; la condition d'adhérence d'un fluide visqueux sur une paroi ne peut pas être vérifiée par une solution de ce type. *H. Cabannes.*

★**Kondo, Jiro. The transient potential of an unsteady supersonic flow.** Proceedings of the Fourth Japan National Congress for Applied Mechanics, 1954, pp. 283–287. Science Council of Japan, Tokyo, 1955.

Il s'agit de l'étude du potentiel des vitesses d'un écoulement non stationnaire linéarisé et supersonique. L'auteur utilise la transformation de Laplace portant simultanément sur le temps t et l'abscisse x mesurée le long de l'axe défini par l'écoulement non perturbé. Quelques applications simples sont envisagées principalement pour des problèmes bidimensionnels; on retrouve ainsi des résultats obtenus précédemment par d'autres voies. L'auteur pense que la méthode qu'il propose est particulièrement indiquée pour l'étude de problèmes posés par le pilotage automatique. *P. Germain* (Paris).

Murgulescu, Elena. Sur quelques problèmes aux limites, de la théorie des mouvements coniques. Com. Acad. R. P. Române 1 (1951), 447–453. (Romanian. Russian and French summaries)

The author considers linearized supersonic conical flow subject to the exact boundary conditions for a flat delta wing with at most one subsonic leading edge. Suppose the wing is at angle of attack χ, is parallel to the y-axis, and has its vertex at the origin. Let the x, y, z velocity components be the real parts of the triple of analytic functions $U(\zeta)$, $V(\zeta)$, $W(\zeta)$; $\zeta=\beta(x+iy)/\{z+[z-\beta^2(x^2+y^2)]^{0.5}\}$; $\beta^2=M^2-1$; 0, 0, W_∞ the undisturbed velocity. Let (*) $G(\zeta)=V(\zeta)\cos\chi+W(\zeta)\sin\chi$, where $\operatorname{Re} G(\zeta)=-W_\infty\sin\chi$ on the image of the wing and on part of $|\zeta|=1$, while $\operatorname{Re} G(\zeta)=0$ on the rest of $|\zeta|=1$. A particular G can be found easily in terms of elementary functions. Then V and W are determined by (*) and $(\zeta^2-1)\beta\, dW/d\zeta=-2i\zeta dV/d\zeta$. The novel feature of the present treatment arises from the fact that if these are to be compatible then $dG/d\zeta=0$ at the roots of (**) $\beta\zeta^2-2i\zeta\tan\chi-\beta=0$. The author removes the logarithmic singularities of V and W at one root of (**) by adding to G an analytic function which does not affect the boundary conditions and has a singularity at the subsonic leading edge or at the intersection of the wing and Mach cone of the vertex. Logarithmic singularities of V and W remain at the other root of (**). *J. Giese* (Aberdeen, Md.).

Murgulescu, Elena. Le problème aux limites de la théorie des mouvements coniques, dans le cas d'une aile mince, de section donnée. Com. Acad. R. P. Române 2 (1952), 489–494. (Romanian. Russian and French summaries)

The authoress considers linearized flow about a thin conical wing subject to simplified boundary conditions applied on one of the coordinate planes instead of the wing. The velocity component normal to this plane is found by quadratures by superposing the solutions of a pair of Signorini integral equations for flow about the mean surface of the wing and the flow at zero incidence about a symmetrical wing with the same thickness distribution. *J. Giese* (Aberdeen, Md.).

Carafoli, E. Sur le caractère hydrodynamique des solutions concernant les mouvements coniques appliquées à la théorie des ailes polygonales. Com. Acad. R. P. Române 2 (1952), 629–644. (Romanian. Russian and French summaries)

Consider linearized supersonic flow about a cone with vertex at the origin of a rectangular coordinate system with X-axis parallel to the undisturbed flow at Mach number M. Let

$$\xi=\eta+i\zeta=B(Y+iZ)/\{X+[X^2-B^2(Y^2+Z^2)]^{0.5}\}$$

where $B^{-2}=M^2-1$ and let u be the X-component of velocity perturbation. Then $u=\operatorname{Re}(u+iu')=\operatorname{Re} f(\xi)$ for some analytic function of ξ. The author considers flows about triangular flat-plate wings with one or two subsonic leading edges; with a subsonic trailing edge and either sub- or supersonic leading edge; and about sets of canted fins composed of pairs of such plates placed antisymmetrically with respect to $\xi=0$. On various segments of the Mach cone $|\xi|=1$, $u=0$ or some other constant. On segments of the η-axis where the boundary condition for the wing is applied u' is constant. For all six examples discussed $|\xi|=1$ and the cone can be mapped onto the real and imaginary axes of a new complex plane by elementary transformations. By interpreting u' as a stream function of an incompressible flow, constant on certain straight segments, and by taking into account the nature of singularities required at leading edges, etc., the author is able to find $f(\xi)$ explicitly in terms of elementary functions by superposing vortex and doublet potentials, etc. *J. Giese* (Aberdeen, Md.).

Carafoli, E. Extension de l'analogie hydrodynamique des mouvements coniques au cas des ailes épaisses et à incidence variable. Com. Acad. R. P. Române 3 (1953), 141–151. (Romanian. Russian and French summaries)

The results of the paper reviewed above are applied first to conical wings composed entirely of plane surfaces, i.e. polyhedral angles. To edges between adjacent faces correspond sources or vortices. Flows about thick polyhedral wings are obtained by the usual superposition of flows about mean surfaces and about symmetrical wings with the same thickness distributions. Then the author considers thick conical wings with continuously turning tangent planes. Roughly speaking these are treated as limits of approximations by polyhedral wings. In this way he approximates the velocity perturbation parallel to the undisturbed flow by an integral over source or vortex layers whose intensities are functions of the local "slope" of the wing. *J. Giese.*

Carafoli, Elie. De la portance et de la résistance de l'aile triangulaire aux vitesses supersoniques. Com. Acad. R. P. Romîne 4 (1954), 261–270. (Romanian. Russian and French summaries)

Let D be a closed curve in the plane $x=1$ which includes a segment of the plane $y=0$ and is symmetrical about $z=0$. Through D pass two cones, the first with vertex at the origin, the second with vertex $(h, 0, 0)$ where $0<h<1$. Now construct a thick wing of triangular plan form whose lower surface lies in $y=0$; the front part of its upper surface is on the first cone; and the rear surface's y-coordinates are the difference between those of both cones. Then construct a wing completely bounded by curved surfaces by fitting together two wings of the sort just described with the triangular plan forms. Flow over

the front surface can be obtained from the thickness distribution as in the preceding paper. The author conjectures that over the rear surfaces, to the accuracy of linearized flow, the perturbation imposed on the front conical flow is virtually the opposite of the perturbations for the conical flow over the second cones at the same undisturbed velocity. Thus the flow over the rear surfaces can also be found from similar integrals of their thickness distributions. Under these assumptions the author computes the drag and lift of the wings. However the reviewer doubts that he has taken proper account of the region of influence of the rear surfaces. *J. Giese.*

Carafoli, E., et Horovitz, B. L'écoulement supersonique autour d'une aile triangulaire à disques marginaux. Com. Acad. R. P. Române **3** (1953), 395–404. (Romanian. Russian and French summaries)

Finston, Morton. A thin wedge in a slightly non-uniform supersonic flow. J. Math. Phys. **34** (1956), 328–334.

Cet article est consacré à l'étude de l'écoulement supersonique autour d'une aile, placée elle même dans le sillage d'une autre aile. Les hypothèses de la linéarisation sont supposées satisfaites; on admet de plus que l'aile influencée est assez loin derrière l'aile influençante pour que le champ des déflexions soit le même dans tous les plans normaux au vent. Après une analyse harmonique du champ des deflexions, on est ramené à résoudre le problème élémentaire suivant: écoulement autour d'une aile d'envergure infinie ondulée placée dans un écoulement supersonique uniforme donné. Ceci conduit à un problème aux limites simple relatif à l'équation des ondes amorties à 2 variables, qui est résolu en appliquant la méthode de Riemann. Le coefficient de pression est le produit d'une fonction sinusoïdale de la variable le long de l'envergure par la fonction de Bessel d'ordre zéro de la variable prise le long de la corde. Une application est faite à l'étude du comportement d'un inclinomètre. *P. Germain* (Paris).

Hayes, Wallace D. Second-order pressure law for two-dimensional compressible flow. J. Aero. Sci. **22** (1955), 284–286.

It is shown that the second-order term for the pressure on a body in subsonic flow is proportional to $\tau^2(1-M^2)^{-\frac{3}{2}}\cdot\{\Gamma M^4+2(1-M^2)\}$, where τ, M and Γ represent thickness ratio (or angle of attack or camber), Mach number, and a thermodynamic parameter that reduces to $(\gamma+1)/2$ for a perfect gas. This result, which corresponds to the Prandtl-Glauert correction for the first-order term, is both simpler and more directly useful than that obtained earlier by Imai [J. Phys. Soc. Japan **9**, 103–108 (1954); MR **15**, 576], who considered the compressibility correction for velocity components only. The author also gives brief consideration to the corresponding result for supersonic flow and to the possible application of the subsonic result to transonic similitude. *J. W. Miles.*

Van Dyke, Milton D. Second-order subsonic airfoil-section theory and its practical application. NACA Tech. Note no. **3390** (1955), 50 pp.

Various methods of obtaining second-order (i.e., including squares of disturbance parameters) solutions for incompressible flow are reviewed. These are corrected for compressibility by the rule of Imai and Hayes [Hayes, British A.R.C. **15, 722, F.M. 1877** (1953); see also the paper reviewed above] and are rendered valid in the neighborhoods of edges by the author's "comparison technique" [NACA Tech. Note no. **3343** (1954)].

Solutions are given for representative airfoils and the results compared with those of other theories and experiment. A scheme for practical, numerical computation is outlined. An appendix tabulates 32 definite integrals that arise in the theory; this tabulation also should be of value in similar, boundary-value problems.

J. W. Miles (Los Angeles, Calif.).

Imai, Isao. The second-order thin airfoil theory for compressible flow. J. Aero. Sci. **22** (1955), 270–271.

Following Van Dyke's work [same J. 21 (1954); 647–648; see also the preceding review], the author has simplified his earlier result for the second-order approximation to the velocity distribution over the airfoil. He also gives the corresponding approximation for the pressure distribution and points out that this result is more general than the similarity rule given by Van Dyke [loc. cit. ante] in that the expression for airfoil ordinate may contain a function that is quadratic in the thickness parameter τ, whereas Van Dyke implicitly assumes simple proportionality between ordinate and τ. An example containing this quadratic term is given.

J. W. Miles (Los Angeles, Calif.).

Wu, T. Yao-Tsu. Two dimensional sink flow of a viscous, heat-conducting, compressible fluid; cylindrical shock waves. Quart. Appl. Math. **13** (1956), 393–418.

L'auteur considère les mouvements plans permanents d'un fluide compressible visqueux et conducteur de chaleur. Les équations dans le plan physique possèdent des solutions indépendantes de l'angle polaire; l'étude des mouvements correspondant fait l'object de ce travail. Des approximations permettent de ramener le problème à une équation différentielle unique entre la vitesse w et la quantité $V=dw/d\,\mathrm{Log}\,r$, où r désigne la distance à un point fixe; les solutions de cette équation sont étudiées en détail et construites. Le retour au plan physique et la comparaison avec la solution non visqueuse fait apparaitre que les effets de viscosité sont importants dans la région transsonique et négligeables ailleurs. L'intégration des équations est ensuite reprise par la méthode dite de Poincaré-Lighthill-Kuo qui consiste à exprimer les fonction inconnues et la variable au moyen d'un paramètre auxiliaire. *H. Cabannes* (Marseille).

Kuo, H. L. On convective instability of a rotating fluid with a horizontal temperature contrast. J. Marine Res. **14** (1955), 14–32.

En linéarisant les équations du mouvement rapportées aux coordonnées cylindriques, et en négligeant un certain nombre des termes, le problème revient à résoudre une équation aux dérivées partielles du 6ème ordre à deux variables (r le rayon vecteur et z l'altitude) a laquelle doit satisfaire la fonction du courant ψ. En imposant certaines conditions aux limites auxquelles doit satisfaire ψ (pour $r=0$, $r=a$, $z=0$, $z=d$), l'auteur essaie de résoudre cette équation par une série double de Fourier. Il s'agit finalement de déterminer les coefficients $A_{m,n}$ de cette série. L'auteur considère deux cas particuliers: 1) une couche mince de liquide d'une très grande extension horizontale: $d\ll a$; 2) une couche dont l'épaisseur est égale à l'extension horizontale: $d=a$. L'auteur croit que les résultats obtenus peuvent avoir des applications pratiques dans les cas de l'atmosphère et de l'océan. *M. Kiveliovitch.*

Mhitaryan, A. M. The unsteady problem of circulation of a monsoon. Akad. Nauk Armyan. SSR. Dokl. **19** (1954), 33–40. (Russian. Armenian summary)

L'auteur génralise les résultats obtenus par Dorodnitzin en introduisant le mouvement à grande échelle, sur une sphère (au lieu de la terre plate) et la force de Coriolis, l'atmosphère étant considérée comme un fluide barocline et visqueux. *M. Kiveliovitch* (Paris).

Mhitaryan, A. M. On the theory of large-scale convection. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **8** (1955), no. 1, 21–40. (Russian. Armenian summary)

L'auteur reprend les formules de son oeuvre analysée ci-dessus et ajoute quelques détails concernant la température du sol ainsi que des calculs de quelques fonctions supplémentaires. *M. Kiveliovitch* (Paris).

Feik, K. Gerichteter Schall. (Ableitung der Kenngrössen.). Hochfrequenztech. Elektroak. **64** (1955), 35–62.

Here is a conveniently arranged collection of formulas for the theoretical characteristics (directivity, gain, etc.) of acoustic radiating elements and arrays. A brief derivation is given for each formula, and an ample bibliography is included. *R. N. Goss* (San Diego, Calif.).

Hunt, Frederick V. Notes on the exact equations governing the propagation of sound in fluids. J. Accoust. Soc. Amer. **27** (1955), 1019–1039.

This clear, careful, and correct expository article is an advance publication from the American Institute of Physics Handbook. It presents topics in exact mathematical theories of waves in fluids which may be of particular interest to specialists in acoustics. The contents are as follows: §§ 1–3,the exact equations of continuum mechanics and of linear viscosity and thermal conduction and their first and second perturbations; §§ 4–5, one-dimensional finite waves according to the material description, including a perturbation procedure following Fubini [Alta Frequenza **4** (1935), 530–581]; § 6, first- and second-order equations for vorticity and streaming; § 7, energy and radiation pressure; § 8, absorption and dispersion due to viscosity, thermal conductivity, and relaxation, following the reviewer [J. Rational Mech. Anal. **2** (1953), 643–741; MR **15**, 757]; § 9, application of the reviewer's results to prove that the sound pressure lags or leads the particle velocity according as $(\gamma-1)Y<1$ or $(\gamma-1)Y>1$, Y being the thermoviscous number; § 10, thermal noise. There are numerous original details, particularly in respect to connection of theory with experiment. For example, the reviewer's results are connected in § 7(e) to second sound in Helium II, in § 7(f) to thunder. *C. A. Truesdell* (Bloomington, Ind.).

Franken, Peter A., and Ingard, Uno. Sound propagation into a moving medium. J. Acoust. Soc. Amer. **28** (1956), 126–127.

Keller, Joseph B. Reflection and transmission of sound by a moving medium. J. Acoust. Soc. Amer. **27** (1955), 1044–1047.

Reflection and transmission coefficients, the angle of refraction, and the phase velocity of the refracted wave are found for the reflection and transmission of sound at a plane interface separating a homogeneous medium at rest from another homogeneous medium in uniform motion. [For a similar treatment of the same problem see the paper listed above]. *P. Chiarulli* (Washington, D.C.).

Friedrichs, K. O., and Keller, Joseph B. Geometrical acoustics. II. Diffraction, reflection, and refraction of a weak spherical or cylindrical shock at a plane interface. J. Appl. Phys. **26** (1955), 961–966.

The geometrical techniques developed in a previous paper by the second author [same J. **25** (1954), 938–947; MR **16**, 761] for obtaining the pressure jump across a weak shock is applied to obtain the pressure jumps across the various shocks arising when a weak spherical (cylindrical) shock produced by an impulsive point (line) source located in a half-space filled with a homogeneous medium intercepts the plane interface between this medium with another homogeneous medium which fills the other half-space. Similar considerations are made for the case of a plane slab (two parallel plane interfaces separating three mediums). Whenever the sound speed ratios are such that total reflection occurs there is no pressure discontinuity across the diffracted wave, but there is such a discontinuity in the pressure derivative and a geometrical technique exactly analogous to the one mentioned previously is obtained and used to compute these derivative jumps across the wave. It is pointed out that these jump expressions for the pressure or pressure derivative are essentially the leading terms of an asymptotic expansion for large ω when instead of an impulsive source the disturbance is caused by a periodic source of angular frequency ω. *P. Chiarulli.*

Brehovskih, L. M., and Ivanov, I. D. On a special form of damping in propagation of waves in stratified nonhomogeneous media. Akust. Ž. **1** (1955), 23–30. (Russian)

Let the velocity of propagation be $c=c_0$ for $0\leq z\leq h$, $c=c_0[1+2a(z-h)]^{-\frac{1}{2}}$ for $z\geq h$, where a is a small parameter. The general theory developed in a memoir of the first author [Izv. Akad. Nauk SSSR. Ser. Fiz. **13** (1949), 505–514, 515–533, 534–545; MR **11**, 563, 564] is applied to the problem of attenuation of the field due to leakage of energy from the homogeneous layer ($0\leq z\leq h$) to the inhomogeneous medium ($z\geq h$) for the case of the above velocity gradient. Estimates of the distance over which the sound field is determined principally by the normal modes, that is the distances over which the layer is effective as a sound channel, are given. *R. N. Goss.*

Carrier, G. F. Sound transmission from a tube with flow. Quart. Appl. Math. **13** (1956), 457–461.

The interior of an open cylinder is $x<0$ and $r<R$, and the steady flow velocities are v_1 in $r<R$ and v_2 in $r>R$. The reflection of an acoustic plane wave at $x=0$ is considered for: I) $v_1=v_2>0$; II) $v_1=v_2<0$; and III) $v_1>0$, $v_2>0$, $v_1\neq v_2$. The formulation and solution of these problems by the Wiener-Hopf technique is presented as an extension of the known solution of the purely acoustic problem ($v_1=v_2=0$) [H. Levine and J. Schwinger, Phys. Rev. (2) **73** (1948), 383–406; MR **9**, 393]. The reflection coefficient in I and II may be related to that obtained by Levine and Schwinger through simple affine transformations, and numerical results are presented. Numerical results for III are not given and would require the numerical evaluation of a rather complicated integral.

J. W. Miles (Los Angeles, Calif.).

See also: Scott, p. 490; Diaz, p. 491; Cabannes, p. 496; Koiter, p. 498; Agostinelli, p. 542; Chandrasekhar and Prendergast, p. 561; Loughhead, p. 561; Nardini, p. 562; Zeulie, p. 562; Carini, p. 562; Kahan, p. 562; Kulikovskiĭ, p. 563; Gibbons, p. 563.

Elasticity, Plasticity

★ **Oshima, Nobunori. A fundamental consideration on the coordinate system in the theory of elasticity.** Proceedings of the Fourth Japan National Congress for Applied Mechanics, 1954, pp. 141–147. Science Council of Japan, Tokyo, 1955.

The author considers the space coordinates x^i of a point P and the space coordinates $'x^i$ of the same point P where $'x^i$ vary with t, the time, and x^i in the following manner:

$$'x^i=x^i, \text{ at time } t;\ 'x^i=x^i+\alpha^i dt, \text{ at time } t+dt$$

and α^i are functions of t, x^i. The relation between the time derivatives $(\partial Q^i\cdot k/\partial t)_{\swarrow}$, $(\partial Q^i\cdot k/\partial t)_{\cdot\swarrow}$ where $Q^i\cdot k$ is a tensor is found to be an extension of the Lie derivative. Further the corresponding relations for the metric tensor, densities, and tensor densities are obtained. Finally, the total derivative of any tensor is determined. From these results, the author obtains the relations between derivatives in three systems: (1) a system at rest (corresponding to the Eulerian view); (2) a system moving with the particles of a material (the so-called Lagrangian view); (3) a system moving in space with velocity $u^i(x^i, t)$. These results are applied to the derivation of the equation of continuity, the acceleration vector and then the equations of motion, the relations between the rate-of-stress and rate-of-strain tensors for an isotropic medium.

N. Coburn (Ann Arbor, Mich.).

Stoppelli, Francesco. Un teorema di esistenza ed unicità relativo alle equazioni dell'elastostatica isoterma per deformazioni finite. Ricerche Mat. 3 (1954), 247–267.

The author presents a proof of existence and uniqueness of solution for the general equations of finite elastic strain for a body with general, not necessarily isotropic strain energy, in the case when the stress vector is prescribed on the boundary. The argument is valid for sufficiently small applied tractions and extrinsic force, under the assumption that the total applied load is equipollent to zero but does not possess an axis of equilibrium. This important paper constitutes the first step toward a general analytic theory for the equations of the classical theory of finite strain and goes far toward establishing the position of the solutions from the linear theory as approximations to certain solutions of the finite theory. Previous attempts at a general theory of integration have rested on formal expansions whose terms may be calculated by solving problems of the linear theory [Signorini, Atti Soc. Ital. Progr. Sci. Riun. XXIV (1936), vol. 3, 6–25; Rivlin and Topakoglu, J. Rational Mech. Anal. 3 (1954), 581–589; MR 16, 88]. The convergence of these expansions has not been proved, and the disturbing question of the possible cumulative effect of the arbitrary infinitesimal rotation introduced at each step has not been mentioned. The author's approach is based on functional analysis and makes no direct use of the above cited previous studies. It is similar in that it considers the applied loads as fixed functions of position multiplied by a parameter θ, and in the course of the proof θ is taken small enough for certain estimates to hold. The presence of this parameter suggests a variational approach. While the author's demonstration is indirect, he remarks that the theorem of functional analysis on which it rests is susceptible of proof by construction, thus in turn implying a convergent procedure of successive approximation for the solution given here.

Because of the extreme formal complexity of the problem, aggravated in the present case by use of the notations of the Italian school, this review attempts to discern the ideas without transcribing the equations.

Let $\mathbf{x}$ be a field of position vectors. A solution of the problem must satisfy three non-linear differential equations of second order in the interior and three non-linear differential equations of first order on the boundary. We may write these symbolically as $T(\mathbf{x})=0$, where T stands for the appropriate six equations. Substitution of any sufficiently smooth $\mathbf{x}$ defines a functional transformation

$$(*) \qquad \mathbf{x}'=T(\mathbf{x})$$

of three-dimensional $\mathbf{x}$'s into six-dimensional $\mathbf{x}'$'s. Existence of solution of the differential equations is then equivalent to inversion of (*) at $\mathbf{x}'=0$. In § 2 the author defines norms in the two spaces in the usual way in terms of bounds for derivatives and Hölder coefficients. Precise sufficient conditions of regularity throughout are stated by the author but will not be repeated in this review.

The proof rests on two devices. The first of these (§ 1 and § 4) is to introduce a constant rotation of the deformed body and the applied loads. Such a rotation does not affect the problem of existence. The author asserts that it is possible to choose this constant rotation in such a way that within a certain neighborhood of $\mathbf{x}=0$ the six-vector $\mathbf{x}'$ is equipollent to 0 in mean over the body [for proof he refers to a previous paper, Rend. Accad. Sci. Fiz. Mat. Napoli (4) 21 (1954), 214–225; MR 17, 96]. On the basis of this he is able to show that (*) is continuous at $\mathbf{x}=0$.

Next (§§ 5–6) the author introduces functional differentials and shows that under his hypotheses of smoothness (*) is continuously differentiable in a neighborhood of $\mathbf{x}=0$. In § 7 he recalls a theorem of functional analysis, according to which a sufficient condition for local invertibility of a continuously differentiable transformation is the complete invertibility of its first equation of variation.

This brings us to the author's second device (§ 8), which is to show that the first variational equation of (*) is identical with the inhomogeneous equations of the classical linear theory of elasticity. In view of the general theorem repeated above, local invertibility of (*) follows. Next the author shows (§ 9) that for sufficiently small θ, the point $\mathbf{x}'=0$ is interior to the neighborhood for which local invertibility has been proved. Thus existence and uniqueness of solution of the equations of the finite theory follow from the existence and uniqueness of sufficiently smooth solutions of the linear theory. The author remarks that there is a gap here in regard to the Hölder exponents of the second derivatives; this special problem for the linear theory he defers until a later work.

The paper concludes (§ 10) by eliminating the constant rotation introduced earlier as a device for the proof.

C. A. Truesdell (Bloomington, Ind.).

Reiner, M. Second-order effects in infinitesimal elasticity. Technion. Israel Inst. Tech. Sci. Publ. 6 (1954/5), 84. (Hebrew summary)

In reference to the usual arguments brought forward in deriving the equations of linear elasticity, the author remarks upon a well-known feature of perturbation of series, namely, that if $\phi=\phi_0+\phi_1+\cdots$, the assumption that ϕ_1 is "small" with respect to ϕ_0 appears to cast doubt upon the validity of the first-order theory for special problems in which ϕ_0 turns out to be rigorously zero. For example, in simple shear the first-order normal stresses are zero but the second-order normal stresses are not. "This reveals a weakness in the foundations of the classical theory of elasticity, and the necessity for a thorough revision."

In the reviewer's opinion, although the usual line of argument is misleading if not faulty, the author's criticism of the linear theory itself is unjustified. The concept of "small" is vague and should be replaced by a limit process as a parameter approaches zero. In the problem of simple shear, the general solution is known, and from it the error of the linear theory is easily shown to approach zero with the angle of shear. For the general boundary-value problem with prescribed stresses, the position of solutions from the linear theory as asymptotic forms of certain solutions from the finite theory has been established by Stoppelli [see the paper reviewed above].

C. A. Truesdell (Bloomington, Ind.).

★**Yoshimura, Yoshimaru. On the natural shearing strain.** Proceedings of the Second Japan National Congress for Applied Mechanics, 1952, pp. 1–4. Science Council of Japan, Tokyo, 1953.

Heinrich, G. Der Energiestrom in elastischen Medien. Österreich. Ing.-Arch. 9 (1955), 148–156.

In the well-known general equation for the rate of change of kinetic and potential energy in a fixed volume, the author names the vector appearing in the surface integral "the energy-transport vector". [Cf. V. Volterra, Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **34** (1899), 366–375; Nuovo Cimento (4) **10** (1899), 337–359.] He obtains the special form assumed by this vector in linear elasticity and discusses some further specializations.

C. A. Truesdell (Bloomington, Ind.).

★**Nishikawa, Jun. A general method for solving two-dimensional problems by means of Fredholm's integral equations.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 169–177. Science Council of Japan, Tokyo, 1952.

$$F(x,y)=\int_0^l \mu(s)\frac{\partial}{\partial s}\left[\{x-\phi(s)\}\tan^{-1}\frac{y-\psi(s)}{x-\phi(s)}\right]ds+$$
$$+\int_0^l \nu(s)\frac{\partial}{\partial s}\left[\{y-\psi(s)\}\tan^{-1}\frac{y-\psi(s)}{x-\phi(s)}\right]ds,$$

where $\mu(s)$, $\nu(s)$ are arbitrary functions, satisfies the biharmonic equation in the region bounded by a simple closed curve $x=\phi(s)$, $y=\psi(s)$ of length l. A pair of integral equations determining $\mu(s)$, $\nu(s)$ is obtained by considering the limiting values of $\partial F/\partial x$, $\partial F/\partial y$ as the interior point x, y approaches the boundary. A method of reducing this pair to a single equation is given. The cases of circular and elliptic boundaries are treated in considerable detail.

R. C. T. Smith (Cambridge, Mass.).

Dyatlovickiĭ, L. I. Reduction of the plane problem of the theory of elasticity with volume forces to a contour problem with arbitrary distribution of volume forces. Inžen. Sb. 21 (1955), 43–60. (Russian)

The author considers problems of plane elasticity in the presence of complicated systems of body forces by replacing these body forces by a discrete distribution of concentrated forces over the region occupied by the body. The stresses due to the body forces may then be removed in an approximate manner and eventually one has to solve a conventional boundary-value problem of plane elasticity. Using for the concentrated forces the expression for an infinite region influence functions may be used which are tabulated for the case of a square with 400 points. The method is applied various plate problems.

J. R. M. Radok (Providence, R.I.).

Sheng, Pei-Lin. Note on the torsional rigidity of cylinders of circular sector cross-sections. Z. Angew. Math. Phys. 6 (1955), 416–419.

Yu, Yi-Yuan. On the complex representation of the general extensional and flexural problems of thin plates and their analogies. J. Franklin Inst. **260** (1955), 269–282.

The author considers problems of generalized plane stress and of flexure of their plates in the presence of general body forces and lateral loads respectively. Using complex-variable techniques he shows that these methods will also apply in the general case when the external forces are described by non-analytic functions. The circular disk is used to illustrate the method.

J. R. M. Radok (Providence, R.I.).

Suhara, Toshiro. On the stresses in laterally loaded plates. Rep. Res. Inst. Appl. Mech. Kyushu Univ. 3 (1955), 180–183.

Szczepaniak, Edmund. Dynamic equations of deformation of the middle surface of a plate subjected to an arbitrary load. Arch. Mech. Stos. 7 (1955), 221–230. (Polish. Russian and English summaries)

Three simultaneous differential equations are derived for the finite deformation of the middle surface of a flexible elastic plate under general dynamic loads of arbitrary direction and finite mass, considering the horizontal components of the displacement. For vertical, mass-less forces and small deflections these equations are reduced to the conventional plate eqaution.

A. M. Freudenthal (New York, N.Y.).

Li, J. P. Bending of a rectangular plate with even and odd order of boundary conditions, Proc. Iowa Acad. Sci. **62** (1955), 384–392.

The title is misleading in that no general principle was presented beyond the well-known Timoshenko's method of superposition for solving clamped rectangular plate problems and that only two specific problems were solved. Essentially the same method of superposition had been used by Huang and Conway [J. Appl. Mech. **19** (1952), 451–460; MR **14**, 516] in solving exactly the same two problems, namely, those of a uniformly loaded rectangular plate with two adjacent edges clamped and the others simply supported or free, the only difference being in the selected form of the deflection function that was superimposed onto that for the corresponding problem of a simply supported plate. The deflection function used in this paper is of the familiar Fourier type. *Y. Y. Yu.*

Dorogobed, A. S. Stress distribution in an orthotropic plate with a circular opening under pure displacement. Inžen. Sb. **21** (1955), 113–119. (Russian)

The author studies the stress distributions in infinite orthotropic plates with a circular hole in pure shear for various orientations of the plate, using the general solution given by S. G. Lehnickiĭ [Anistropic plates, OGIZ, Moscow-Leningrad, 1947; MR **10**, 415].

J. R. M. Radok (Providence, R.I.).

Naghdi, P. M. The effect of elliptic holes on the bending of thick plates. J. Appl. Mech. **22**, 89–94 (1955).

The present paper considers stress concentration problems in transverse bending of thin plates, with the effect of transverse shear deformation taken into account. In extension of earlier work of the reviewer for plates with circular holes, and utilizing the basic theory of this earlier work [same J. **12** (1945), A-69–A-77; MR **7**, 42], the author considers plane bending and pure twisting of a plate with elliptical hole. As a step towards the solution of these problems, he constructs suitable particular solutions involving Mathieu functions. The satisfaction of the boundary conditions of the problem then will involve the solution of an infinite system of algebraic equations for an infinite number of series coefficients. In his numerical work the author limits himself to one term in the series solution which means that the boundary conditions are not accurately satisfied, and it is difficult to assess the significance of the numerical results which are given in the paper.

E. Reissner (Cambridge, Mass.).

Johnson, Millard W., and Reissner, Eric. On inextensional deformations of shallow elastic shells. J. Math. Phys. **34** (1956), 335–346.

In the present paper the classical problem of inextensional deformations of thin elastic shells is considered from the point of view of the linear theory of shallow shells. The basic equations of the theory of shallow shells are taken in the form in which they were obtained by K. Marguerre [Proc. 5th Internat. Congress Appl. Mech., Cambridge, Mass., 1938, Wiley, New York, 1939, pp. 93–101]. The main concern of the paper are two questions: (i) the form of the boundary conditions along a free edge of a shallow shell which deforms inextensionally; (ii) the determination of the frequencies of free inextensional vibrations of shallow shells.

Let $z=z(x, y)$ be the equation of the middle surface of the shell, $D=D(x, y)$ its bending stiffness factor, $C=C(x, y)$ its longitudinal stiffness factor and ν Poisson's ratio of the shell material. Let $w=w(x, y)$ be the transverse deflection of points of the middle surface, $F=F(x, y)$ an Airy stress function for stress resultants tangential to the middle surface and $p=p(x, y)$ a transverse load function. In terms of the foregoing quantities, and assuming an isotropic material with linear stress-strain relations, the differential equations of the theory may be reduced to the following system of two simultaneous equations for F and w,

$$(1)\quad \nabla^2(C^{-1}\nabla^2 F)+(1+\nu)[2C^{-1}{}_{xy}F_{xy}-C^{-1}{}_{xx}F_{yy}-C^{-1}{}_{yy}F_{xx}] = 2z_{xy}w_{xy}-z_{xx}w_{yy}-z_{yy}w_{xx},$$

$$(2)\quad \nabla^2(D\nabla^2 w)+(1-\nu)[2D_{xy}w_{xy}-D_{xx}w_{yy}-D_{yy}w_{xx}] = p(x, y)-2z_{xy}F_{xy}+z_{xx}F_{yy}+z_{yy}F_{xx}.$$

Subscripts in (1) and (2) indicate partial differentiation, and ∇^2 is the two-dimensional Laplace operator. The case of a homogeneous shell of uniform thickness is obtained from (1) and (2) by considering C and D as constants.

The assumption of inextensibility of the middle surface of the shell is introduced into the theory by assuming an infinitely large value of the longitudinal stiffness factor C. Accordingly, the differential equations of the linear theory of inextensional deformations of thin shallow isotropic shells are equation (2) together with the following abbreviated form of equation (1):

$$(3)\quad 2z_{xy}w_{xy}-z_{xx}w_{yy}-z_{yy}w_{xx}=0.$$

As noted by the authors, in general the assumption of inextensibility reduces a system of two simultaneous fourth-order to two successive second-order equations. Accordingly, while for finite C there are four boundary conditions to be satisfied along the edge of a shell only two boundary conditions are expected for the exceptional case $C=\infty$.

A separate consideration is necessary for the case of a flat plate for which $z=c_0+c_1x+c_2y$. For this case the two differential equations (1) and (2) are uncoupled and equation (2) remains a fourth-order equation for w, even when $C=\infty$. This means that while small transverse bending of flat plates occurs without middle-surface extension it is not properly a limiting case of inextensional deformations of shells. Stated in another way, an interchange of the order of the limiting processes 1) $z=c_0+c_1x+c_2y$, 2) $C=\infty$ is of influence on the final result obtained by means of the limiting operations.

R. Gran Olsson (Trondheim).

Volterra, E. The equations of motion for curved elastic bars deduced by the use of the "method of internal constraints." Ing.-Arch. **23** (1955), 402–409.

The equations of motion for curved bars are derived taking into account the influence of shear and of rotatory inertia by assuming that during motion the sections originally normal to the axis of the bar remain plane.

The elastic isotropic homogeneous bar is supposed to be an elongated solid with a curvilinear baricentric axis. It is referred to a fixed system of orthogonal cartesian coordinates $(0, \xi, \eta, \zeta)$ and to a system of curvilinear right-handed coordinate system $(0, x, y, z)$ in which the origin coincides with a generic point on the central axis of the bar, x with the central axis (and has as positive direction the positive direction of the central axis), y coincides with the principal normal (n) and is directed positively towards the center of curvature, and z coincides with the binormal (b) at the origin. The condition that the sections originally normal to the axis of the bar remain plane during motion implies that the elastic displacement vector V at a generic point P of the bar depends linearly only on the coordinates y and z or that the following vector equation be verified:

$$(1)\quad V(x, y, z, t)=\nu(x, t)+y\lambda(x, t)+z\mu(x, t),$$

where ν, λ and μ are three vectors, a priori unknown, functions of the variables x and t.

This method of analysis, which was used more than twenty years ago to study static deformations of curved elastic bars [E. Volterra, Verh. Internat. Math.-Kongresses, Zürich, 1932, Bd. II, Füssli, Zürich-Leipzig, 1933, pp. 247–248; Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **16** (1932), 220–222; **20** (1934), 424–428, 463–467; **21** (1935), 14–19; **23** (1936), 329–332] is referred to as the "method of internal constraints", since the fundamental hypothesis on which the method is based rests on the assumption that the elastic displace-

ment of the bar occurs under the special condition of internal constraint expressed by equation (1). The geometric significiance of the internal constraint has been first discussed, then the expression for the square of the element of length (ds^2) has been derived in the more general case in which the central axis of the bar has both first and second curvatures. Thus, from the fundamental (Euclidean) metric tensor it is possible to calculate the components of the strain and then the expressions for the potential and kinetic energies of the bar in terms of the nine (unknown) components ν_x, ν_y, ν_z; $\lambda_x, \lambda_y, \lambda_z$; μ_x, μ_y, μ_z of the three vectors ν, λ and μ.

The application of Hamilton's Principle furnishes the nine differential equations and the boundary conditions which uniquely determine the nine unknown functions ν_x, ν_y, ν_z; $\lambda_x, \lambda_y, \lambda_z$; μ_x, μ_y, μ_z in the case of free vibrations. The case in which external forces are acting on the bar can be dealt with by introducing into the variational principle the work done by these external forces.

Although the method of analysis presented in this paper applies to the general case in which the curvature of the central axis and the inertia characteristics of the cross-section of the bar are known functions of the variable x, for simplicity's sake the equations of motion have been derived only for the particular case in which the curvature and cross-section area of the bar are constant.

R. Gran Olsson (Trondheim).

Volterra, E. A one-dimensional theory of wave-propagation in elastic rods based on the "method of internal constraints." Ing.-Arch. 23 (1955), 410–420.

The equations of motion which were derived in the preceding paper are used. These equations were obtained by assuming that during motion the sections originally normal to the axis of the bar (x axis) remain plane, or that the following vector equation be verified:

$$(1) \qquad V(x, y, z; t)=\nu(x, t)+y\lambda(x, t)+z\mu(x, t).$$

In equation (1) V represents the elastic displacement vector at a generic point P of the bar, ν, λ and μ are three vectors, a priori unknown, functions of the coordinate x and of the time t. The bar is referred to a system of orthogonal cartesian coordinates $(0, x, y, z)$ where x coincides with the axis of the bar, and y and z with the principal axes of inertia of the cross-section of the bar.

The results obtained in the particular case of lateral vibrations of bars show an encouraging agreement of the results given by the approximate theory based on the assumption of internal constraints and the exact theory derived from the use of the equations of the mathematical theory of elasticity.

This approximate theory is in the dynamic case completely contained in the constraint equation (1) and in the application of Hamilton's Principle. Accordingly the concept of the Timoshenko shear coefficients is not used. In the general case of wave propagation in elastic straight rods this theory unifies a number of separate engineering treatments of the problem. Moreover, the same theory can be applied to the study of vibrations of curved bars taking into account the effects of shear and rotatory inertia, as has been shown in the preceding paper. The mathematical simplicity of the theory and its degree of accuracy justify its use in dealing with engineering problems in vibrations of curved or straight bars for which more exact theories cannot be used because of their mathematical complexities. *R. Gran Olsson.*

Vlasov, V. Z. The method of initial functions in problems of elasticity theory. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 7, 49–69. (Russian)

The author proposes use of the displacements u_i and of the stresses τ_{i3} as dependent variables for solving three-dimensional problems of elasticity. The remaining stresses may then be determined in terms of these quantities and one has to find solutions of six differential equations satisfying certain boundary conditions. Expressing the general solutions in the form of Maclaurin series for x_3 and using differential operators which either have the form of infinite series or of formal sums of these series, these solutions lead to functional relationships between the dependent variables and their values on the plane $x_3=0$. The method is applied to thick plates of constant and variable thickness subject to surface tractions.

J. R. M. Radok (Providence, R.I.).

Nowiński, J., and Turski, S. Solution of the Lamé problem for a heterogeneous cylinder by means of the ARR differential equation analyser. Arch. Mech. Stos. 7 (1955), 419–424. (Polish. Russian and English summaries)

Nowiński, J., Olszak, W., and Urbanowski, W. On thermoelastic problems in the case of a body of an arbitrary type of curvilinear orthotropy. Arch. Mech. Stos. 7 (1955), 247–265. (Polish. Russian and English summaries)

Thermal stresses in the isotropic-elastic continuum due to a non-linear temperature distribution are obtained by the superposition of a hydrostatic pressure due to restrained expansion on the stresses produced by fictitious body forces proportional to the temperature gradients and surfaces forces proportional to the temperature. In the paper the expressions for the body and surfaces forces are derived for orthogonal curvilinear problems of mechanical and thermal orthotropy in the directions of the coordinate system. *A. M. Freudenthal* (New York, N.Y.).

Handelman, George H. A note on the transverse vibration of a tube containing flowing fluid. Quart. Appl. Math. 13 (1955), 326–330.

"The problem of determining the lowest frequency of vibration of a tube containing flowing fluid has recently received considerable attention. Long [J. Appl. Mech. 22 (1955), 65–68] has considered the tube as a beam and has calculated the frequencies for various end conditions by a power-series method. He has pointed out that care must be taken in evaluating the resulting high-order determinants to avoid erroneous results. Niordson [Kungl. Tekn. Högsk. Handl. Stockholm no. 73 (1953); MR 15, 582] has given a treatment of the problem based on shell theory and has derived the beam equation as one approximation. Furthermore, for the case of simply supported ends, he has given closed analytic solutions for two limiting cases of the parameters and numerical results for other values. It is the purpose of this note to show that it is possible to determine the nature of the frequencies for the various end conditions used by Long for two ranges of the flow velocity solely from the structure of the differential equation without determining specific solutions." (From author's Introduction.)

G. W. Morgan (Providence, R.I.).

Kopeĭkin, Yu. D., and Leonov, M. Ya. On a special case of buckling of a compressed bar. Prikl. Mat. Meh. **19** (1955), 736–737. (Russian)

Deĭneko, K. S., and Leonov, M. Ya. A dynamic method of investigating stability of a compressed bar. Prikl. Mat. Meh. **19** (1955), 738–744. (Russian)

Mandel, Jean. Sur le fluage du corps de Maxwell. C. R. Acad. Sci. Paris **241** (1955), 1550–1552.

The tensor equations for the general Maxwellian body are indicated and solved in terms of the initial and boundary conditions. It is shown that the body has a discrete spectrum of relaxation times. *B. Gross.*

Prager, William. The theory of plasticity: A survey of recent achievements. Proc. Inst. Mech. Engrs. **169** (1955), 41–57.

★**Kondo, Kazuo. On the geometrical and physical foundations of the theory of yielding.** Proceedings of the Second Japan National Congress for Applied Mechanics, 1952, pp. 41–47. Science Council of Japan, Tokyo, 1953.

This papers contains a qualitative discussion of the geometric interpretations of some physical phenomena. First, dislocation theory is discussed. The author's interpretation of this theory depends on a "Burger's vector" which measures the torsion of a non-Riemannian space. Elastic deformations are distinguished from plastic deformations by the condition that the topology is preserved in the former but not in the latter. That is, the author notes that the initial and final coordinates of a plastic material cannot be related by a unique point transformation since the final coordinates depend upon the history of the deformation and thus involve non-holonomic quantities. Sharp yielding is said to be realized by mappings involving Riemann spaces. The author states that an alternative scheme is to study the possibility of immersing a Riemann space in a Euclidean space. Again, the author suggests that since plastic effects depend on the past history of the material, a probability theory approach may be of value. Finally, some remarks are made as to small deformations. [The reviewer does not follow the author's arguments on sharp yielding and the interpretation of plastic effects by probability.]

N. Coburn (Ann Arbor, Mich.).

Moskvitin, V. V. Elastic-plastic deformations of bodies under repeated loading. Prikl. Mat. Meh. **19** (1955), 714–724. (Russian)

Moment-curvature relations are obtained for cyclic loading of a beam of rectangular cross-section. The analysis is based on a general total-strain theory for small elastic-plastic deformations, specialized to the case of an elastic-linearly strain hardening material. All strains are supposed to vary in proportion. Repeated loading of a thick, hollow sphere under internal pressure is treated on the same basis. *R. M. Haythornthwaite.*

Abbasov, A. A., and Mirzadžanzade, A. H. Approximate solution of a problem on unsteady motion of a viscous-plastic medium in a circular cylindrical tube. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 12, 122–124. (Russian)

See also: Koiter, p. 498; Gross, p. 542.

MATHEMATICAL PHYSICS

★**Margenau, Henry, and Murphy, George Moseley. The mathematics of physics and chemistry.** D. Van Nostrand Company, Inc., Princeton, N.J., 1956. xii+604 pp. $7.95.

Although minor changes have been made and known errors corrected, this edition remains essentially like the first one in plan [1943; MR **4**, 268].

Kilmister, C. W. The anlysis of observations. II. Quart. J. Math. Oxford Ser. (2) **6** (1955), 161–172.

[For part I see Bastin and Kilmister, Proc. Roy. Soc. London. Ser. A. **212** (1952), 559–576; MR **14**, 227.] The author gives a formal discussion of various ideas intended to clarify Eddinton's work on wave-tensor calculus.

A. H. Taub (Urbana, Ill.).

Proca, A. Particules de très grandes vitesses en mécanique spinorielle. Nuovo Cimento (10) **2** (1955), 962–971.

The author discusses solutions of the equations

$$\frac{d\xi}{d\tau}=-\lambda_\varrho\gamma^\varrho\xi,\quad \frac{d\xi^+}{d\tau}=\lambda_\varrho\xi^+\gamma^\varrho,\quad \frac{dx^\varrho}{d\tau}=\xi^+\gamma^\varrho\xi,\quad \frac{d\lambda_\varrho}{d\tau}=0,$$

where λ_ϱ is a constant four-vector, ξ is a four component spinor, τ is a parameter representing proper time, γ^ϱ are the Dirac matrices and ξ^+ is the spinor adjoint to ξ. It is shown that if $dx^\varrho/d\tau$ is a null-vector, λ_ϱ must also be one. Various tensors built up from the spinor ξ are discussed.

A. H. Taub (Urbana, Ill.).

Proca, A. Interférences en mécanique spinorielle. Nuovo Cimento (10) **2** (1955), 972–979.

The author generalizes the spinor defined in the preceding review to be a spinor field in space-time. Various tensor equations satisfied by quantities defined in terms of this spinor field are discussed. *A. H. Taub.*

Gürsey, Feza. Dual invariance of Maxwell's tensor. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **19** (1954), 154–160 (1955). (Turkish summary)

It is shown that the Maxwell stress-energy tensor is invariant under the transformation

$$f_{\mu\nu}\to f_{\mu\nu}'=f_{\mu\nu}\cos\theta+\tilde{f}_{\mu\nu}\sin\theta,$$

where $f_{\mu\nu}$ is the Maxwell field tensor and $\tilde{f}_{\mu\nu}$ is the dual tensor. *A. H. Taub* (Urbana, Ill.).

Optics, Electromagnetic Theory, Circuits

v. Fragstein, C. Ist eine Lichtbewegung stets umkehrbar? Opt. Acta **2** (1955), 16–22.

The question of reversibility of a light path is considered for propagation involving absorbing media. In general, the transmission coefficients for propagation across a boundary between two absorbing media and the corresponding phase changes are different for incidence in opposite directions. Some comparison formulae are derived. It is also shown that for some metals there exists an angle of incidence for which a wave penetrating

into the metal from air does not change its direction, whilst no such special direction exists for propagation into air from the metal. For other metals the opposite is found to be the case. *E. Wolf* (Manchester).

Jurek, Bohumil. Les surfaces sphériques dans les systèmes à deux miroirs. Rozpravy Československé Akad. Věd. **65** (1955), no. 10, 1–25. (Czech. Russian and French summaries)

In order that a centred combination of two mirrors should be axially stigmatic it is in general necessary that one of the mirrors be aspheric. If the combination is to be aplanatic, both mirrors must in general be aspheric. The paper is concerned with finding the special conditions under which axial stigmatism may be achieved with two spherical mirrors, and aplanatism with one spherical and one aspheric mirror. It is found that in the first case the object must be at the centre of curvature of one or both of the mirrors. This system is also aplanatic. The only other combination (with a real image) that appears practicable and which contains a spherical mirror is the well known sphere-cardioid system used in microscope condensors. Readers interested in these questions should also consult papers by C. R. Burch [Proc. Phys. Soc. **59** (1947), 41–46, 47–49]. *E. Wolf* (Manchester).

Sternberg, Robert L. Successive approximation and expansion methods in the numerical design of microwave dielectric lenses. J. Math. Phys. **34** (1956), 209–235.

It is required to find the form of a zoned lens of revolution having in a meridian plane two foci at a finite and two at an infinite distance. The lens diameter, index of refraction, distance to the finite foci, directions to the infinite foci, and the wave length are all given parameters. Using the theorem of Malus, the author obtains power-series expansions for the function which describes the lens surface. The arguments are mainly heuristic, and questions of uniqueness and convergence are left unsettled except insofar as one can appeal to experimental data, which are in good agreement, with the author's numerical results. *R. N. Goss* (San Diego, Calif.).

★**Avazashvili, D. Z. Three dimensional problem of the diffraction of monochromatic electromagnetic waves.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 6 pp. $3.00.

Translated from Doklady Akad. Nauk SSSR (N.S.) **82** (1952), 29–32; MR **13**, 801.

★**Brekhovskikh, L. M. Diffraction of waves from an irregular surface. I. General theory.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 27 pp. $15.00.

Translated from Ž. Eksper. Teoret. Fiz. **23** (1952), 275–288; MR **14**, 703.

★**Brekhovskikh, L. M. Diffraction of waves by irregular surfaces. II. Application of the general theory.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 21 pp. $11.00.

Translated from Ž. Eksper. Teoret. Fiz. **23** (1952), 289–305; MR **14**, 703.

★**Fok, V. A. Diffraction of radiowaves around the earth's surface.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 80 pp. $7.50.

This is the translation of a monograph [Diffrakciya radiovoln vokrug zemnoĭ poverhnosti, Izdat. Akad. Nauk SSSR, Moscow, 1946] giving a detailed presentation of the results summarized in an earlier paper by the author [C.R. (Dokl.) Acad. Sci. URSS (N.S.) **46** (1945), 310–313; MR **7**, 100]. The most novel part of the monograph is the approximation in the penumbral region. This is given either as a contour integral or as an infinite series involving the function

$$w(t)=\left(\frac{-\pi t}{3}\right)^{1/2}e^{2\pi i/3}H_{1/3}^{(1)}[\tfrac{2}{3}(-t)^{3/2}]=u(t)+iv(t)$$

[a multiple of $\mathrm{Ai}(te^{2\pi i/3})$]. 4–5S tables of $u(t)$, $u'(t)$, $v(t)$, $v'(t)$ with first differences are given for $t=-9(.02)9$.

A. Erdélyi (Pasadena, Calif.).

★**Fok, V. A. Theory of radiowave propagation in an inhomogeneous atmosphere for a raised source.** Translated by Morris D. Friedman, 2 Pine St., West Concord, Mass., 1955. 38 pp. $15.00.

Translated from Izv. Akad. Nauk SSSR **14** (1950), 71–94; MR **12**, 225.

Tao, Kazuhiko. On the propagation of ultra short waves beyond the horizon. Jap. J. Geophys. **1** (1954), 27–79 (1 plate).

With the accumulation of evidence which has rendered untenable the classical line-of-sight theory of propagation of short radio waves beyond the horizon, two major rival theories have arisen to explain the invariably larger fields observed: the theory of turbulent scattering, and the internal reflection theory based upon some assumed gradient of the index of refraction. The present paper is a contribution to the latter. The model used is a spherical earth surrounded by a lower atmosphere (which we call layer 1) and an inversion layer of finite thickness. The gradient is thus trilinear. Within each layer the formula of Eckersley is used for the refractive index. The formal problem is then to find the field in layer 1 of a radiating vertical magnetic dipole, also in layer 1, subject to appropriate continuity conditions at the interfaces. As in problems of this kind, the chief task is to bring the enormously complicated coefficients resulting from the formal solution into such a form that the results may be evaluated and tested. To this end the author employs Watson's classical method of residues.

In a second part of the paper results of field-strength measurements under varying meteorological conditions at a Japanese test station are presented and correlated with the preceding theory. The same problem has been dealt with in a report of T. J. Carroll and R. M. Ring [Normal tropospheric propagation of short radio waves well beyond the horizon, Lincoln Laboratory, Mass. Inst. of Technology, Tech. Rep. **38** (1954)] and in more recent unpublished work by Carroll. *R. N. Goss.*

de Hoop, A. T. Variational formulation of two-dimensional diffraction problems with application to diffraction by a slit. Nederl. Akad. Wetensch. Proc. Ser. B. **58** (1955), 401–411.

This paper deals with the diffraction of time-harmonic electromagnetic waves by an aperture in a perfectly conducting plane screen of vanishing thickness. Complex wave functions are used with time-factor $\exp(-i\omega t)$. The screen lies in the plane $z=0$, and the boundaries of the aperture (which may be disjoint) are all parallel to the axis of y. If the incident plane waves are such that only

the y-component of either the electric or magnetic vector is non-zero, the problem can be expressed as a two-dimensional scalar one.

It is shown that the complex amplitude of the distant diffracted field can be written in a stationary form of the Levine-Schwinger type. And this variational principle is applied to the diffraction of plane waves by an infinite slit of finite width $2b$. The aperture distribution, the complex amplitude of the distant diffracted field and the transmission coefficient are determined up to and including the terms of relative order $(kb)^4$, where $2\pi/k$ is the wave length. *E. T. Copson* (St. Andrews).

Teisseyre, R. The diffraction on a conducting wedge. The general solutions for dipole field. Nuovo Cimento (10) **2** (1955), 869–871.

The author generalizes the method of Senior [Quart. J. Mech. Appl. Math. **6** (1953), 101–114; MR **14**, 933] for solving the diffraction problem of a dipole field in the presence of a conducting half-plane to cover the case of a conducting wedge. Details are promised elsewhere.

A. E. Heins (Pittsburgh, Pa.).

Pachner, Jaroslav. A theory of the diffraction of a scalar wave by a plane screen. Czechoslovak J. Phys. **5** (1955), 305–339. (Russian summary)

This paper studies the effect of an aperture in a screen subject to excitation by a scalar plane. The author divides the solution into two parts, one of which is the customary Kirchhoff approximation while the other takes care of the boundary conditions on the screen. The case of long wave-lengths is discussed in detail.

A. E. Heins (Pittsburgh, Pa.).

Wait, James R. On the scattering of spherical waves by a cylindrical object. Appl. Sci. Res. B. **4** (1955), 464–468.

The author investigates the relation between plane wave and spherical wave scattering from a cylindrical object. *A. E. Heins* (Pittsburgh, Pa.).

Bouix, Maurice. La polarisation elliptique du rayonnement électromagnétique. Ann. Télécommun. **9** (1954), 275–281, 298–304, 345–351.

Le but de la suite des trois articles réunis sous ce titre est de mettre sous une forme adaptée à l'étude du rayonnement en polarisation elliptique des notions bien connues, mais qui jusqu'ici ont été présentées dans les textes surtout en vue d'une utilisation en polarisation rectiligne.

From the author's summary.

Carini, Giovanni. Le onde elettromagnetiche nei conduttori in moto. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 41–47.

It is shown that when a plane electromagnetic wave is propagating in a conductor which itself is moving in the direction of propagation, there is a simple approximate relation between the index of refraction and the dielectric constant, provided the conductivity is large.

R. N. Goss (San Diego, Calif.).

Zin, Giovanni. L'analisi dei cavi coassiali irregolari. Univ. e Politec. Torino. Rend Sem. Mat. **12** (1953), 127–143.

In this paper the author gives a method for determining the irregularity of a co-axial cable from the measured data of the input impedance taken over a frequency band. The procedure is roughly as follows. An integral expression for the input impedance in terms of the reflection coefficient of the line has been derived. This is based on the assumption that the characteristic impedance separates into a product of a function of position and a function of frequency, and the irregularity extends only over a finite interval (distance) of the semi-infinite line. By considering a new co-axial line composed of the actual line plus a finite uniform extention, the author has obtained a Laplace transformation between the characteristic capacitance (object function) and characteristic input impedance (image function) of the co-axial cable. By expressing the capacitance in a Fourier series and in virtue of the finite extension of the irregularity, the coefficients in the Fourier series can be calculated by evaluating an integral continuing the impedances over a frequency band. *N. Chako* (New York, N.Y.).

Poincelot, Paul. La répartition du courant le long d'une antenne cylindrique à l'émission. Ann. Télécommun. **10** (1955), 186–194, 219–228.

Četaev, D. N. On a point source of direct current in a nonhomogeneous half-space. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1955**, 265–266. (Russian)

Solution of the problem of distribution of a direct current in the half-space $z>0$ of known conductivity $\sigma(z)$ is obtained by determining the potential $U(z, r)$. If

$$U(z, r)=\int_0^\infty J_0(rt)\cdot u(z, t)dt,$$

the equation

$$\sigma(z)\cdot\frac{\partial^2 u}{\partial z^2}+\sigma'(z)\frac{\partial u}{\partial z}-\lambda^2\sigma(z)\cdot u=0$$

is integrated. *E. Kogbetliantz* (New York, N.Y.).

Horváth, János. The equation of motion of the electron. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 411–450. (Hungarian)

The paper treats the problem of determining the equation of motion of an electron in an external electromagnetic field, taking into account the "self-field" of the electron due to its charge and magnetic moment. The work is carried out entirely in the framework of classical electrodynamics. The assumption of a point-electron is made, which of course leads to divergencies. According to the author, however, these divergencies are purely mathematical in nature and are not results of physical difficulties. In order to eliminate these divergencies, the author makes use of a method of M. Riesz [C.R. Congrès Internat. Math., Oslo, 1936, t. II, Brøgger, Oslo, 1937, pp. 44–45; Acta Math. **81** (1949), 1–223; MR **10**, 713] to solve the second-order hyperbolic partial differential equation for the four-potential of the problem, which in turn is obtained from the Lagrangian by a variational method. The equations of motion of the electron are also calculated from the Lagrangian with the help of a variational method. These and the expression for the self-force are thus obtained in terms of the energy-momentum tensors of the external field and the self-field, and the latter is then obtained by the Riesz method which in effect selects out the "finite part" of the divergent expressions. The appendix gives an outline of the Riesz method. The problem discussed in the paper has also been treated by Dirac [Proc. Roy. Soc. London. Ser. A. **167** (1938), 148–169] and by Bhabba and Corben [ibid. **178** (1941), 273–

314; MR 3, 158] and in fact the latter authors obtain essentially all the results of the present paper but by a more tedious (although perhaps physically more transparent) method. The present paper does not discuss the physical meaning, validity, and limitation of this classical renormalization process. *M. J. Moravcsik.*

Carini, Giovanni. Sulle equazioni di Minkowski per i conduttori in moto. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 152–158.

From the Minkowski equations of the electromagnetic field in a moving conductor, a second-order differential equation satisfied by the vectors **E**, **D**, **B**, **H** is deduced. It agrees with a result of G. Lampariello [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1954), 222–228; MR **17**, 216]. *R. N. Goss* (San Diego, Calif.).

Chandrasekhar, S. On force-free magnetic fields. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 1–5.

A force-free magnetic field satisfies the conditions

$$\operatorname{div}\mathbf{H}=0$$

and

$$\mathbf{H}\times\operatorname{curl}\mathbf{H}=0 \text{ or } \operatorname{curl}\mathbf{H}=\alpha\mathbf{H}.$$

For the case when α is constant, the author derives explicit formulae for the most general axially symmetric force-free field. He also discusses the case when α is a function of position.

Any axially symmetric field can be expressed as the sum of a toroidal and a poloidal component, in the form

$$\mathbf{H}=\mathbf{1}_z\times\mathbf{r}T+\operatorname{curl}(\mathbf{1}_z\times\mathbf{r}P),$$

an expression that satisfies the first condition identically. (Here $\mathbf{1}_z$ is a unit vector parallel to the axis of symmetry, $\mathbf{r}$ is the position vector, and T and P are any two functions that do not depend on the azimuthal angle.) Of the three remaining conditions, only two are independent, and these take the form (in the case when α is constant)

$$T=\alpha P \text{ and } \Delta_5 P=-\alpha T,$$

where

$$\Delta_5\equiv\frac{\partial^2}{\partial\varpi^2}+\frac{3}{\varpi}\frac{\partial}{\partial\varpi}+\frac{\partial^2}{\partial z^2}$$

$$\equiv\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{(\partial\cos\theta)^2}-\frac{4\cos\theta}{r^2}\frac{\partial}{\partial\cos\theta}$$

in cylindrical and spherical polar coordinates respectively. Thus the function P can be any linear combination of the fundamental solutions

$$P_n=r^{-3/2}C_{n+3/2}(\alpha r)C_r^{3/2}(\cos\theta),$$

where $C_n^{3/2}(\cos\theta)$ is a Gegenbauer polynomial, and $C_{n+3/2}$ is a linear combination of the Bessel functions $J_{n+3/2}$ and $J_{-(n+3/2)}$.

With the help of these formulae, the author obtains three further results: (i) If α is discontinuous on the surface of a sphere of radius R centered on the origin, αR must be one of the zeros of the cylinder function $C_{n+3/2}$. (ii) The toroidal and poloidal components of the field contribute equally to the total magnetic energy (for which the author derives a simple expression). (iii) The equations determining the lines of force have the integral

$$r^{1/2}C_{n+3/2}(\alpha r)\sin^2\theta C_n^{3/2}(\cos\theta)=\text{constant}.$$

Essentially the same problem has been treated previously by Lüst and Schlüter [Z. Astrophys. **34** (1954), 263–282; MR **17**, 110], who did not, however, succeed in obtaining an explicit general solution. *D. Layzer.*

Chandrasekhar, S., and Prendergast, Kevin H. The equilibrium of magnetic stars. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 5–9.

This paper treats a generalization of the problem considered in the preceding paper. If a magnetic field exists in an axially symmetric, perfectly conducting, mass of fluid in hydrostatic equilibrium under its own gravitation, the non-azimuthal components of the quantity curl ($\mathbf{H}\times$ curl $\mathbf{H}$) must vanish. (In the preceding paper the condition was $\mathbf{H}\times\operatorname{curl}\mathbf{H}=0$.) The authors show that these conditions will be satisfied if and only if

$$\tilde{\omega}^2T=F(\tilde{\omega}^2P),$$

where F is an arbitrary function (see the preceding paper for notation). If the fluid is incompressible, all three components of $\operatorname{curl}(\mathbf{H}\times\operatorname{curl}\mathbf{H})$ must vanish, and the authors derive the stronger condition

$$\Delta_5 P=-T\frac{d}{d(\tilde{\omega}^2P)}(\tilde{\omega}^2T)+\Phi(\tilde{\omega}^2P),$$

where Φ is an arbitrary function. The authors point out that these considerations solve the problem of finding the conditions that must be satisfied by the fluid velocity in the stationary flow of an inviscid, incompressible fluid, since under these conditions

$$\operatorname{curl}(\mathbf{u}+\operatorname{curl}\mathbf{u})=0=\operatorname{div}\mathbf{u}.$$

Previous attempts to solve the problems considered in this paper have been made by Ferraro [Astrophys. J. **119** (1954), 407–412; MR **16**, 183], G. Gjellestad [ibid. **119** (1954), 14–33; MR **15**, 748], P. H. Roberts [ibid. **122** (1955), 508–512], and Lüst and Schlüter [Z. Angew. Math. Mech. **35** (1955), 45–47; MR **16**, 1169]. *D. Layzer* (Cambridge, Mass.).

Loughhead, R. E. Hydromagnetic stability of a current layer. Austral. J. Phys. **8** (1955), 319–328.

This paper is devoted to the consideration of the following two problems: In an infinite plane slab of density ϱ_1 confined between $-a\leq x\leq+a$ a uniform magnetic field (H_z) in the z-direction prevails. The field is zero outside this slab while the density is ϱ_2. Considering solutions of the relevant linearized hydromagnetic equations of the form $\exp i(\omega t+my+nz)$ appropriate for small oscillations, the author shows that the two modes of hydromagnetic waves are propagated with the velocities

$$V_1=\pm\left(1+\frac{\varrho_2}{\varrho_1}\tanh\varkappa a\right)^{-\frac{1}{2}}V_a,$$

and

$$V_2=\pm\left(1+\frac{\varrho_2}{\varrho_1}\coth\varkappa a\right)^{-\frac{1}{2}}V_a$$

where $V_a=H_z/(4\pi\varrho_1)^{\frac{1}{2}}$ is the Alfvén velocity and $\varkappa=(m^2+n^2)^{\frac{1}{2}}$.

The second problem considered is when a uniform current flows in the slab in the z-direction and produces a field $H_y=Ax$ (where A is a constant) in the y-direction. It is further assumed that outside the slab $H_y=A_+$ for $x>a$ and $H_y=A_-$ for $x<a$; also, that the z-component of the impressed magnetic field is B for $-a\leq x\leq a$, B_+ for $x>a$, and B_- for $x<a$. To investigate the stability of this situation, solutions of Maxwell's equations are sought which are stationary but whose dependence on the y-

and the z-directions is given by $\exp i(my+nz)$. With suitable boundary conditions, the criterion for stability is obtained in the form

$$(1)\quad e^{4\pi\alpha}=\frac{[(\varkappa a+\gamma+1)-\alpha^2(\varkappa a+\gamma)][(\varkappa a-\gamma+1)+\beta^2(\varkappa a-\gamma)]}{[(\varkappa a+\gamma-1)+\alpha^2(\varkappa a+\gamma)][(\varkappa a-\gamma-1)-\beta^2(\varkappa a-\gamma)]},$$

where $\gamma=\varkappa nB/Am$

$$\alpha=\frac{mA_++nB_+}{mAa+nB}\text{ and }\beta=\frac{mA_-+nB_-}{-mAa+nB}.$$

If the impressed magnetic field outside the slab is zero, $\alpha=\beta=0$ and equation (1) reduces to

$$(2)\quad e^{4\pi\alpha}=\frac{(\varkappa a+1)^2-\gamma^2}{(\varkappa a-1)^2-\gamma^2}.$$

From this equation it follows that the initial situation is unstable since it allows just exactly one positive root. The other special case considered is when $H_y=\eta x/a$, in the slab and $H_y=\pm\varepsilon\eta$ (where ε is a constant) outside the slab and the magnetic field in the z-direction is everywhere zero. In this case ($\gamma=0$) equation (1) becomes

$$(3)\quad e^{4\pi\alpha}=\frac{[(1+\varepsilon^2)\varkappa a+1][(1-\varepsilon^2)\varkappa a+1]}{[(1+\varepsilon^2)\varkappa a-1][(1-\varepsilon^2)\varkappa a-1]}.$$

From this equation it follows that the situation is stable if $\varepsilon>1$ while it is unstable if $0\leqq\varepsilon\leqq1$.

S. Chandrasekhar (Williams Bay, Wis.).

Loughhead, R. E. Eigen oscillations of compressible, ionized fluids. Austral. J. Phys. 8 (1955), 416–418.

The radial oscillation of a right circular cylinder of constant density in which a uniform axial magnetic field of intensity H prevails is considered. It is shown that the circular frequencies of oscillation are given by

$$\omega_n=\frac{j_n}{R}\left(\frac{H^2}{4\pi\varrho}+c^2\right)^{\frac{1}{2}},$$

where c denotes the velocity of sound, ϱ the density, R the radius of the cylinder and j_n is a zero of the Bessel function $J_1(x)$. *S. Chandrasekhar* (Williams Bay, Wis.).

Nardini, Renato. Sulla convezione del calore in un liquido elettricamente conduttore soggetto ad un campo magnetico. Ann. Univ. Ferrara. Sez. VII. (N.S.) 3 (1954), 93–98.

The equations of heat conduction and hydromagnetics [in the so-called Boussinesq-approximation; cf. Chandrasekhar, Phil. Mag. (7) **43** (1952), 501–532; MR **14**, 813] which govern an incompressible electrically conducting fluid in the presence of an external magnetic field are considered. It is shown that subject to the boundary conditions, the velocity on the bounding surface is zero, and the temperature, the tangential component of the magnetic field and the normal component of the temperature gradient are maintained at specified values, the solution of the equations is unique. *S. Chandrasekhar.*

Zeuli, Tino. Propagazione di oscillazioni magneto-idrodinamiche in un fluido elettricamente conduttore che riempie un semispazio. Boll. Un. Mat. Ital. (3) **10** (1955), 23–31.

The hydromagnetic equations appropriate for a viscous compressible fluid of finite electrical conductivity are considered. It is supposed that gravity ($-gz$) acts in the z-direction, a uniform magnetic field with components in the x- and the z-directions prevails and that the fluid is moving with constant velocity, v_0, in the y-direction. Perturbations about this stationary solution are considered and it is shown that the corresponding linearized equations allow solutions with a time and space dependence of the form $\exp(in\omega t-\alpha_n{}^2z)$, provided a certain characteristic equation of order four in $\alpha_n{}^2$ (and involving the other parameters of the problem) is satisfied. The various resulting modes are briefly discussed. [The reviewer is of the opinion that the solution obtained by the author is based on a physically inconsistent model: For, the stationary state from which the analysis proceeds is a medium of uniform density even though gravity prevails; the author allows for this by making the pressure increase with z ($p=\varrho gz$); this is clearly inadmissible. The propagation of hydromagnetic waves in a stratified atmosphere in a gravitational field has been correctly solved by Ferraro, Astrophys. J. **119** (1954), 393–406; MR **15**, 761.]

S. Chandrasekhar (Williams Bay, Wis.).

Carini, Giovanni. Osservazioni riguardanti le onde magneto-idrodinamiche di Alfvén. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18(87)** (1954), 150–156.

By retaining the terms in the displacement current in the hydromagnetic equations (appropriate for an inviscid, incompressible fluid of infinite electrical conductivity), the author shows that the velocity of propagation of a plane wave is given by

$$V=cH_0\left(\frac{\mu}{4\pi\varrho c^2+\varepsilon\mu^2H_0{}^2}\right)^{\frac{1}{2}},$$

where H_0 is the intensity of a prevailing uniform magnetic field, ϱ the density, c the velocity of light and ε and μ are the dielectric constant and the coefficient of magnetic permeability, respectively. Pure electro-magnetic waves and Alfvén's hydromagnetic waves are obtained as limiting cases of this formula. *S. Chandrasekhar.*

Carini, Giovanni. Propagazione di onde piane magneto-idrodinamiche in un liquido conduttore mobile in un campo magnetico omogeneo. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18(87)** (1954), 439–444.

The analysis of the previous papers is generalized to include the effect of a finite electrical conductivity σ. In this case the waves are damped and the first order change in the velocity of wave-propagation is given by $-(\varepsilon\beta^2V^2\omega^2/4\sigma^2\mu^2H_0{}^4)$, where V is the velocity quoted in the preceding review and ω is the circular frequency of the wave. Also the distance in which the wave is damped by a factor e is given by $z_0=2\sigma\mu^{5/2}H_0{}^3(\varrho+\varepsilon\mu^2H_0{}^2)^{1/2}/\varrho^2\omega^2$.

S. Chandrasekhar (Williams Bay, Wis.).

Carini, Giovanni. Condizioni di compatibilità dinamica nella teoria delle onde magneto-idrodinamiche. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18(87)** (1954), 433–438.

It is shown that in the limit of zero amplitude a disturbance in an inviscid, incompressible fluid of infinite electrical conductivity will be propagated with the usual Alfvén velocity. *S. Chandrasekhar.*

Kahan, Théo. Sur l'électrodynamique des milieux ionisés turbulents. C. R. Acad. Sci. Paris **241** (1955), 1726–1727.

The author suggests that analogous to the notion of the spectrum of the mean square kinetic energy which has proved so useful in ordinary hydrodynamic turbulence, one should introduce the spectrum of the magnetic energy when one considers hydromagnetic turbulence.

S. Chandrasekhar (Williams Bay, Wis.).

Kulikovskiĭ, A. G. On the decay of homogeneous isotropic magneto-hydrodynamic turbulence in an incompressible fluid. Prikl. Mat. Meh. **19** (1955), 551–556. (Russian)

The equations of hydromagnetic turbulence as written by Chandrasekhar [Proc. Roy. Soc. London. Ser. A. **204** (1951), 435–449; MR **14**, 424] are considered for the case when the electrical conductivity is infinite and the triple correlation $\langle h_i h_j u_k' \rangle_{av}$ dominates $\langle u_i u_j u_k \rangle_{av}$. $H_i = h_i(4\pi\varrho/\mu)^{\frac{1}{2}}$ denotes the components of the magnetic intensity, u_i the components of the velocity and the primed and the unprimed quantities refer to the field variables at two points separated by a vector ζ_i and a distance r. The corresponding equations are

$$(1) \qquad \frac{\partial Q}{\partial t} = -2\left(r\frac{\partial}{\partial r}+5\right)S + 2\nu\left(\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}\right)Q,$$

$$(2) \qquad \frac{\partial H}{\partial t} = 2P,$$

where Q, H, S and P are the defining scalars of the correlations

$\langle u_i u_j' \rangle_{av}$, $\langle h_i h' \rangle_{av}$, $\langle h_i h_j u_k' \rangle_{av}$ and $\langle (h_i u_j - h_j u_i) u_k' \rangle_{av}$.

The behavior of these scalars near the origin is given by

$$Q = -Q_0(t) + Q_2(t)r^2 + \cdots; \quad H = -H(t_0) + H_2(t)r^2 + \cdots;$$
$$S = S_0(t) + S_2(t)r^2 + \cdots; \quad P = 5S_0(t) + P_2(t)r^2 + \cdots.$$

Also $dQ_0/dt = 10S_0 - 10\nu Q_2$ and $dH_0/dt = -10S_0$. The author investigates the circumstances under which Q, H, S and P are expressible in the forms

$$Q = -Q_0 q(x), \quad H = -H_0 h(x), \quad S = S_0 s(x) \text{ and } P = 5S_0 p(x),$$

where $x = r/l$ and l is a function of t only. The author shows that such solutions are permissible if S_0 is a constant,

$$Q = H_0(C_1 H_0^{\alpha_1 - 1} + \gamma)$$

and

$$\frac{dH_0}{dt} = -\frac{a_1 \nu}{C_2^2}\,\frac{H_0^{(2\alpha_1+\alpha_1+1)}}{AH_0^{\alpha_1} + H_0 b/C_1},$$

where C_1, C_2, γ, a_1, A and b are constants. There are certain restrictions on these constants; one of these is $\alpha_1 \geqq 1$. The problem also admits the special solution: $H_0/Q_0 = \text{const.}$; $H = H_0 - 10S_0 t$, and $l = l_0[Q_0(t)]^{\frac{1}{2}}$.

S. Chandrasekhar (Williams Bay, Wis.).

Gibbons, J. J. Some exact solutions of the Lorentz invariant problem of the motion of two electric fluids. Canad. J. Phys. **33** (1955), 819–823.

Simple exact solutions of Maxwell's equations, plus Lorentz equation, for charged, pressure-free, non-viscous fluids. *F. A. E. Pirani* (London).

Bouwkamp, C. J. A simple method of calculating electrostatic capacity. Institute of Engineering Research, University of California, Berkeley, Series No. **60**, Issue No. 138, i+6 pp. (1955).

The following theorem is proved and exemplified: Let C denote the capacity with respect to infinity of a bounded conductor, with surface S, in free space. Let R denote any sphere with centre M and radius a, and let S' be the image of S with respect to R in the sense of inversion by reciprocal radii. Then $C = a^2 V_0$, in which V_0 is the potential at M due to the charges induced on the conductor S' by a negative unit point charge located at M when S' is connected to earth. *L. M. Milne-Thomson.*

Fel'dbaum, A. A. On the design of optimal systems by means of phase space. Avtomat. i Telemeh. **16** (1955), 129–149. (Russian)

Let C denote a specified part of a regulating system (usually the motor) and let $X(t)$ be the variable to be regulated, $X_0(t)$ its regulation value at time t. The problem is to design an optimal regulator Y characterized by the condition that the transient error $x = X_0 - X$ reach the value 0 and stay at the value 0 within a time τ that is minimal. In a previous paper [Avtomat. Telemeh. **14** (1953), 712–728] the author proved that if C is such that it imposes on $X(t)$ the restriction $|v| = |a_0 X^{(n)} + a_1 X^{(n+1)} + \cdots + a_n X| \leq M$ with $a_0 p^n + \cdots + a_n p = 0$ having n real roots $p_i \leq 0$ then Y will be optimal only if $v(t) = M\sigma(t)$ where $\sigma(t) = +1$, -1 alternatingly in n consecutive intervals. The optimal regulator Y can be synthesized from the function F for which

$$v(t) = F\{x(t), \cdots, x^{(n-1)}(t), X_0(t), \cdots, X_0^{(n-1)}(t)\}.$$

In this paper the author discusses how, for given X_0, $\cdots$, $X_0^{(n-1)}$ the surface S can be constructed which separates the regions $F > 0$, $F < 0$ in the phase space of points $(x, \cdots, x^{(n-1)})$. He proves that this surface is independent of X_0, $\cdots$, $X_0^{(n-1)}$ ("stationary") if the limiting condition of C is $|a_0 X^{(n)} + \cdots + a_{n-q} X^{(n-q)}| \leq M$ and $X_0(t)$ is a polynomial of degree $< n-q$. For example, in the case $n=3$, $q=0$ the function

$$F = M \operatorname{sign}\left\{x + \frac{z^3}{3M^2} + \operatorname{sign} w\left[\frac{yz}{M} + \frac{1}{M^{\frac{1}{2}}}\left(\frac{z^2}{2M} + y \operatorname{sign} w\right)^{3/2}\right]\right\},$$

where $y = x'$, $z = x''$ and $w = y + (z^2/2M) \operatorname{sign} z$.

M. Golomb (Lafayette, Ind.).

Şaichin, A. Solutions périodiques de certaines équations différentielles dans la théorie des phénomènes paramétriques. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 667–682. (Romanian. Russian and French summaries)

In an electrical circuit the inductance varies with time according to the law $L(1 - m\cos\Omega t)$, the impressed electromotoric force is sinusoidal (frequency ω). The solution is expanded in a trigonometric series (frequencies $\omega + k\Omega$, $k = 0, \pm 1, 2\pm, \cdots$), and numerical and graphical results are given for the case of resonance. [This problem has been treated by several authors, among them the reviewer, Hochfrequenztech. Elektroak. **46** (1935), 73–77; Arch. Elektrotech. **29** (1935), 473–489.] *A. Erdélyi.*

Gadžiev, M. Yu. On the characteristic frequencies of some linear systems. Izv. Akad. Nauk Azerbaĭdžan. SSR **1954**, no. 6, 3–13. (Russian. Azerbaijanian summary)

Characteristic frequencies of linear systems with inductance, capacitance and an element such as a vacuum tube are determined by a graphical procedure.

N. Levinson (Cambridge, Mass.).

Chakrabarty, Nirmal Baran. Synthesis of a network for a prescribed time function. Indian J. Phys. **28** (1954), 473–484.

Two methods of synthesis are given. I. A relation is established between the coefficients of the power-series expansion of the transfer immitance $g(p)$ and the transforms of the moments of the time function (output) $f(t)$. The power series of $g(p)$ is considered as a recurring series of order n. This allows determination of the coefficients of denominator $D(p)$ and nominator $N(p)$ of the polynomial

$N(p)/D(p)$ that defines $g(p)$. Knowledge of them permits construction of the network. II. $f(t)$ is represented by its time series (sequence of values at equal intervals of time). Then the free modes of the system are determined a) by a spectrum method and b) from the known property of the time series to be linear auto-regressive. *B. Gross.*

Kac, A. M. Determination of the parameters of a regulator from a given characteristic equation of the control system. Avtomat. i Telemeh. **16** (1955), 269–272. (Russian)

If $K_1(p)=A(p)/B(p)$ is the operator of the load and $K_2(p)=X(p)/Y(p)$ is the operator of the regulator then $F(p)\equiv B(p)Y(p)-A(p)X(p)=0$ is the characteristic equation of the control system. The author describes the well known algorithm by which the polynomials $X(p)$, $Y(p)$ are determined from the given polynomials $A(p)$, $B(p)$, $F(p)$. *M. Golomb* (Lafayette, Ind.).

Piesch, Johanna. Die Matrix in der Schaltungsalgebra zur Planung relaisgesteuerter Netzwerke. Arch. Elek. Übertr. **9** (1955), 460–468.

An exposition of the application of Boolean matrices to the analysis of combinational relay switching circuits with more than two terminals is presented in this paper. The procedure followed is similar to that developed by Hohn and Schissler in a previous paper [Bell System Tech. J. **34** (1955), 177–202; MR **17**, 436]. Several examples showing how contacts can be saved are given. *C. Y. Lee* (Chatham, N.J.).

See also: Oberhettinger, p. 476; Berghuis, p. 476; Letov, p. 487; Davies, p. 496; Chu and Churchill, p. 539; Brehovskih and Ivanov, p. 553; Proca, p. 558.

Quantum Mechanics

Hill, E. L. Relativistic theory of discrete momentum space and discrete space-time. Phys. Rev. (2) **100** (1955), 1780–1783.

This paper describes a theory, which is based on the postulate that the momentum and energy of a particle can have only those values, for which they transform like the components of a four-vector under Lorentz transformations with rational coefficients. This leads to a model of discrete momentum-energy space with no lower or upper bound to the values of the momentum of the particle. It is also found possible to express the quantum-mechanical wave function of a particle as an infinite series in terms of almost periodic functions instead of the usual periodic functions. However, the resulting formalism involves serious difficulties. For, the almost periodic functions cannot be made to vanish at infinity unless they vanish everywhere, and the usual treatment for the calculation of interactions between different fields breaks down in the present formalism. The author also briefly discusses a model of discrete space-time along similar lines. *S. N. Gupta* (Lafayette, Ind.).

Zaĭcev, G. A. Real spinors in curvilinear coordinates and in pseudoriemannian space. Ž. Eksper. Teoret. Fiz. **29** (1955), 345–353. (Russian)

The calculus of real spinors developed by the author in earlier papers [cf. same Ž. **29** (1955), 166–175; MR **17**, 330] is here extended first to curvilinear coordinate-systems and second to curved space-time. The definitions are made so that the spinor ψ behaves like a scalar under differentiation, thus the covariant derivative of ψ is simply $(\partial\psi/\partial x^\alpha)$, while the matrices R^α behave like a vector. The usual definitions, following the work of V. A. Fock [Z. Physik **57** (1929), 261–277], distribute the effects of differentiation between ψ and R^α in a different way. The difference is not essential, because it is always possible to change from one representation to the other by a local spin-transformation. The author deduces the Dirac equation from his formalism by a heuristic argument. *F. J. Dyson* (Princeton, N.J.).

Tietz, T. Über die Abzählung der Eigenwerte der Schrödinger-Gleichung im begrenzten Gebiete. Acta Phys. Acad. Sci. Hungar. **5** (1955), 347–352.

In certain cases, for example that of an electron in a Coulomb field, the solutions of the Schrödinger equation involve confluent or Gaussian hypergeometric functions; and the number of eigenvalues on a bounded interval may be determined from known properties of the real zeros of hypergeometric functions. *N. D. Kazarinoff.*

Reulos, René. Application à la Mécanique ondulatoire d'une nouvelle transformation de la relativité restreinte. C. R. Acad. Sci. Paris **241** (1955), 1723–1726.

Infeld, L., and Plebański, J. On a certain class of unitary transformations. Acta Phys. Polon. **14** (1955), 41–75. (Russian summary)

It is known that the commutation relation $[\xi_1, \xi_2]=i$ cannot be satisfied by bounded Hermitean operators in Hilbert space, but that under certain assumptions there is an irreducible representation of the relation by unbounded ξ_1 and ξ_2, unique up to unitary equivalence [J. von Neumann, Math. Ann. **104** (1931), 570–578]. On the other hand, it is known [F. Rellich, Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys. Chem. Abt. **1946**, 107–115; MR **9**, 192] that without some assumptions this uniqueness theorem does not hold; other inequivalent "pathological" representations exist. The paper under review tacitly assumes that the uniqueness theorem holds and solves the following problems. Let

$$\bar{\xi}_\alpha=\sum_{\beta=1}^{2} t_\alpha{}^\beta \xi_\beta + t_\alpha, \tag{1}$$

where $t_\alpha{}^\beta$ and t_α are real, and $t_\alpha{}^\beta$ has determinant one. Then the $\bar{\xi}_\alpha$ will also satisfy the commutation relations and

$$U\xi_\alpha U^{-1}=\bar{\xi}_\alpha, \tag{2}$$

where U is some unitary operator. Find U explicitly in terms of $t_\alpha{}^\beta$ and t_α and the ξ_β. Typically, U is expressed as $\exp T$, where T is a quadratic form in the ξ_α. Let

$$X_1=\tfrac{1}{2}(\xi_1{}^2+\xi_2{}^2),\quad X_2=\tfrac{1}{2}(\xi_2{}^2-\xi_1{}^2),\quad X_3=\tfrac{1}{2}(\xi_1\xi_2+\xi_2\xi_1).$$

The transformations (1) induce transformations of the X_h $(h=1, 2, 3)$. The set of these latter transformations forms a group isomorphic to the homogeneous real Lorentz group in three-dimensional space-time. Find canonical forms for general homogeneous quadratic forms in ξ_1 and ξ_2. Explicit formulae are given for a variety of transformations U of use in quantum mechanics. *A. S. Wightman* (Princeton, N.J.).

Hamilton, J. Centre-of-mass state vectors. Proc. Cambridge Philos. Soc. **52** (1956), 97–106.

Relativistic center-of-mass systems are discussed. The eigenvectors of the total linear momentum associated with

eigenvalue zero (center-of-mass state vectors) in the single-time formalism of quantum field theory are expanded in spherical harmonics. These expansions are given explicitly for the following systems of two particles of definite total angular momentum: two scalar mesons, two nucleons, and the meson-nucleon system. The question of pseudoscalar interaction is briefly considered, especially with regard to meson-nucleon scattering.

F. Rohrlich (Iowa City, Ia.).

Bergmann, Peter G., and Goldberg, Irwin. Dirac bracket transformations in phase space. Phys. Rev. (2) **98** (1955), 531–538.

Dirac [Canad. J. Math. **2** (1950), 129–148; MR **13**, 306] has constructed a new type of bracket $(A, B)^* = (A, B) + (A, C^a)(B, C^b)F_{ab}$, where (A, B) is the Poisson bracket, C^a (in Dirac's terminology) are "second class" constraints, and the matrix F_{ab} is defined from $F_{ab}(C^b, C^c) = \delta_a^c$. In the present paper the authors show that the Dirac bracket represents the commutator of infinitesimal transformations in phase space which are not canonical but form a group in that they are the only transformations that preserve the form of all the constraints of a theory as well as the canonical form of the equations of motion. The invariant sub-group of this group corresponds to coordinate transformations, gauge transformations and the like. All members of the original group have generators but the generators of the invariant sub-group are zero. The authors propose to reformulate the quantization problem for covariant quantum theories in terms of these generators and the Dirac brackets between them, thus avoiding the difficulties connected with the constraints.

A. Salam (Cambridge, England).

Case, K. M. Hamiltonian form of integral spin wave equations. Phys. Rev. (2) **100** (1955), 1513–1514.

Schrödinger [Proc. Roy. Soc. London. Ser. A. **299** (1955), 39–43; MR **17**, 219] and the author [Phys. Rev. (2) **99** (1955), 1572–1573; MR **17**, 442] have recently obtained the Hamiltonian forms of the spin-one and spin-zero wave equations, using a special representation of the Duffin-Kemmer matrices β_μ. These are rederived using only the algebraic properties of the matrices. *A. Salam.*

Nakai, Shinzo. Point transformation and its application. Progr. Theoret. Phys. **13** (1955), 380–388.

For a certain class of point transformations, it is shown that the transformed field-theoretic Lagrangian directly gives the same Hamiltonian as is obtained by a unitary transformation on the Hamiltonian derived from the original Lagrangian. The results are used to rederive the Foldy transformation [Phys. Rev. (2) **84** (1951), 168].

A. Salam (Cambridge, England).

Hayashi, Chushiro. Hamiltonian formalism in non-local field theories. Progr. Theoret. Phys. **10** (1953), 533–548.

Papers like those of P. Kristensen and C. Møller [Danske Vid. Selsk. Mat.-Fys. Medd. **27** (1952), no. 7] and of C. Bloch [ibid. **27** (1952), no. 8] have created the impression that the Hamiltonian formalism cannot be used in non-local field theories and that only the S matrix, which is not explicitly concerned with the intermediate stages of the reaction, is of any significance. This erroneous impression has been attacked by several Japanese workers such as Y. Katayama [Progr. Theoret. Phys. **9** (1953), 561–562; **10** (1953), 31–56; MR **15**, 187] H. Umezawa and Y. Takahashi [ibid. **9** (1953), 501–523; MR **15**, 186], and the present authors. These investigations have shown that a Hamiltonian in the interaction representation can also be defined in the case of non-local interactions, if one uses the Yang-Feldman method [Phys. Rev. (2) **79** (1950), 972–978; MR **12**, 569]. This method gives solutions of the field equations in terms of integrals over space and time and hence they can be used to calculate the S matrix directly, without reference to the Hamiltonian formalism. The method, however, can also yield a Hamiltonian which in fact can be useful in establishing the unitary character of the S matrix. The present authors, for instance, claim that the S matrix defined by Kristensen and Møller in the article cited above is not unitary. In the present paper an interaction Hamiltonian (which is non-unique) is expressed as an infinite power series in the coupling constant. Explicit calculations are carried out up to and including the fourth order, and they agree with Kristensen and Møller except in the fourth order. The integrability condition for the equation of motion is satisfied by any two points which lie on a space-like surface or, unlike in local field theories, on a surface which is time-like in a finite domain of the four-dimensional space. The relationship of the present formalism to the theory of Pauli [Nuovo Cimento (9) **10** (1953), 648–667; MR **15**, 81] and to the extension by Gell-Mann and Low [Phys. Rev. (2) **84**, (1951), 350–354; MR **13**, 413] of the Bethe-Salpeter equation is discussed and a divergence-free treatment of bound-state problems using the present results is predicted but not demonstrated.

M. J. Moravcsik (Upton, N.Y.).

Medvedev, B. V. On the unitarity of the S-matrix in the quantum theory of a field with nonlocal interaction. Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 433–435. (Russian)

It was found by C. Hayashi [see the paper reviewed above] that the S-matrix of a field-theory with a conventional extended-source interaction becomes non-unitary when calculated by perturbation theory as far as terms in g^4. The author here states that the non-unitary term in g^4 arises only from the unobservable vacuum-current which was included in Hayashi's Lagrangian. If the Lagrangian is correctly anti-symmetrized to remove the vacuum-current effect, the non-unitary term also becomes identically zero. *F. J. Dyson* (Princeton, N.J.).

Abrikosov, A. A., and Halatnikov, I. M. The use of two limiting momenta in field theories. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 993–996. (Russian)

The results of L. D. Landau, A. A. Abrikosov and I. M. Halatnikov [same Dokl. (N.S.) **95** (1954), 497–500; MR **16**, 315] on the asymptotic behavior of Green's functions in quantum field theory are here re-derived, using a modified form of high-momentum cut-off. Instead of a single cut-off momentum Λ, the authors use one cut-off Λ_b for the Bose field and another Λ_f for the Fermi field. The results are the same as before. *F. J. Dyson.*

Pomerančuk, I. Equality to zero of the renormalized charge in quantum electrodynamics. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 1005–1008. (Russian)

The title of this paper is intended to imply that quantum electrodynamics with point interactions is inconsistent for any non-zero value of the physical electron charge e.

What is in fact proved is that a particular form of cut-off theory with two cut-off momenta [see the paper reviewed above] does not tend to a reasonable limit as the cut-off momenta tend to infinity. The argument is quite simple. It is proved that in the cut-off theory, when the two cut-offs are related by the inequality

$$\log(\Lambda_p^2/\Lambda_k^2) \gg 1,$$

the only effect which goes to infinity with $\log \Lambda_p^2$ is the charge-renormalization; all other effects are cut down by the smaller cut-off Λ_k^2 and can be made as small we please by increasing Λ_p^2 sufficiently. The relation between e and the unrenormalized charge e_1 is then

$$e^2 = e_1^2[1+(e_1^2/3\pi)\log(\Lambda_p^2/m^2)]^{-1},$$

a result which (because of the double cut-off) holds asymptotically in the limit $\Lambda_p \to \infty$ and is independent of perturbation theory. When $\Lambda_p \to \infty$, this implies $e^2=0$, irrespective of the behavior of e_1.

The reviewer believes that the question, whether or not quantum electrodynamics is consistent, is still open. The author shows that one possible cut-off theory does not lead to a consistent limit as the cut-off tends to infinity; this is an interesting result but does not settle the question, because many other limiting processes might be used and might lead to consistent limits. The question of consistency cannot be adequately discussed without also raising the question of uniqueness. *F. J. Dyson.*

Gor'kov, L. P., and Halatnikov, I. M. Perturbation theory and asymptotic behavior of Green's functions in the electrodynamics of particles with spin zero. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 799–802. (Russian)

The asymptotic behavior of Green's functions in spin-zero electrodynamics is investigated by the method which was used by L. D. Landau, A. A. Abrikosov, and I. M. Halatnikov [same Dokl. (N.S.) **95** (1954), 497–500; MR **16**, 315] for the electrodynamics of particles with spin $\frac{1}{2}$. The Kemmer matrix form of the theory is employed. The results are similar to those of the spin-$\frac{1}{2}$ theory; for example, the photon Green's function has the asymptotic form

$$d_t(k^2) = [1+(e_1^2/12\pi)\log(-k^2/\Lambda^2)]^{-1},$$

the only difference from the spin-$\frac{1}{2}$ theory being the change from 3 to 12 in the numerical coefficient.

F. J. Dyson (Princeton, N.J.).

Gor'kov, L. P., and Halatnikov, I. M. Asymptotic behavior of Green's functions in the electrodynamics of particles with spin zero. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 197–200. (Russian)

In the paper reviewed above the authors studied spin-zero electrodynamics with no special choice of gauge. Here it is shown that the earlier results are obtained more simply, and the whole theory comes into a much closer correspondence with spin-$\frac{1}{2}$ electrodynamics, when the gauge is chosen in such a way that the transverse and longitudinal photon Green's functions are equal. This choice of gauge means that the total photon Green's function becomes simply

$$D_{\mu\nu}(k^2) = D(k^2)\delta_{\mu\nu};$$

in other words it is the Feynman gauge. *F. J. Dyson.*

Alder, Kurt, and Winther, Aage. Matrix elements between states in the Coulomb field. Danske Vid. Selsk. Mat.-Fys. Medd. **29**, no. 18, 18 pp. (1955).

Adler, Kurt, and Winther, Aage. On the exact evaluation of the Coulomb excitation. Danske Vid. Selsk. Mat.-Fys. Medd. **29** (1955), no. 19, 20 pp.

In the first paper it is shown that the matrix elements between radial eigensolutions in a Coulomb field can be expressed as Laplace integrals involving products of confluent hypergeometric functions. These integrals can be evaluated in terms of hypergeometric functions of two variables. Recurrence relations are discussed both in the non-relativistic and in the relativistic case, and specializations to discrete-discrete, discrete-continuous, and continuous-continuous transitions are given.

In the second paper, the computation of the total cross-section for Coulomb excitation is reduced to the computation of the matrix elements between radial eigenfunctions, and the results of the first paper are used to investigate recurrence formulas and limiting cases. Numerical results are also given. *A. Erdélyi.*

Heber, G. Zu Tomonagas Methode der Behandlung mittelstark gekoppelter Felder. Methodische Studien. I, II. Ann. Physik (6) **15** (1955), 157–173, 174–177; Berichtigung **16** (1955), 192.

Tomonaga's intermediate coupling approximation [Progr. Theoret. Phys. **2** (1947), 6–24] is rederived for the case of neutral scalar meson field coupled to static sources of finite extension (source density function). The method consists essentially in an application of Ritz's variational procedure to quantum field theory. The state vector of the system is expanded in Fock state vectors (associated with a definite number n of mesons) and the set of expansion coefficients $\{c_n\}$ is determined which minimizes the total energy. The extension of this method to unquantized sources in nonrelativistic motion leads to an equation for $\{c_n\}$ which the author could solve only in perturbation theory treating terms in n^2 small compared to terms in n. The same difficulty arises for quantized scalar non-relativistic sources. However, within this approximation a scalar source surrounded by a scalar meson cloud has a finite (non-zero) binding energy. An apparent explicit dependence of the total energy on the normalization volume is shown to disappear for any fixed wave-packet structure of the source particle.

F. Rohrlich (Iowa City, Ia.).

Heber, G. Zur Theorie der Elementarteilchen. I. Ann. Physik (6) **16** (1955), 43–51.

The intermediate coupling approximation of an earlier paper [see the preceding review] is applied to a determination of the source structure. A non-relativistic scalar "bare nucleon" (represented by a wave packet) is the source and interacts with an unlimited number of relativistic neutral scalar "mesons". The energy of the system ΔW contains the interaction energy U as well as the kinetic energy of the wave packet, E_0. Minimizing ΔW yields a non-linear integral equation for the normalized momentum distribution $b(\mathbf{p})$ which characterizes the wave packet. Since this equation is difficult to solve, the author assumes $b(\mathbf{p})$ to be of the form $b(\mathbf{p})=a$ for $|\mathbf{p}| \leqq L/2$, $b(\mathbf{p})=0$ for $|\mathbf{p}|>L/2$, where the cut-off momentum L is to be determined by minimizing ΔW. It is then found that ΔW has a minimum at $L \neq 0$ (i.e. $\Delta W \neq 0$) only for coupling constants $g^2/\hbar c > 26$. A typical case ($g^2/\hbar c \sim 50$) leads to a spatial extension of the wave packet of the

order of the proton Compton wave length and to $\Delta W_{\min}=-0.12\,Mc^2$, M being the proton mass. Similar results are to be expected for any spherically symmetrical distribution $b(\mathfrak{p})$ provided the cut-off is finite. It is noted that for fixed sources ($E_0=0$) $\Delta W_{\min}=-\infty$ corresponding to $L=\infty$ and a point source. The method does not seem to give reasonable results when applied to electrodynamics. *F. Rohrlich* (Iowa City, Ia.).

Werle, J. Singular potentials in relativistic equations of motion. Acta Phys. Polon. **14** (1955), 233–249. (Russian summary)

A uniform method is proposed for obtaining "adiabatic" potentials from relativistic equations of motion for nucleons in non-quantized meson fields. It is shown that for small r, repulsive forces do appear in the scalar and the pseudoscalar potentials. Likewise a strong repulsive term appears in the "equivalent" potential between two nucleons at short distances, in agreement with the results of the much more elaborate "non-adiabatic" calculations [M. Levy, Phys. Rev. (2) **88** (1952), 725–739]. *A. Salam* (Cambridge, England).

Karlson, E. The eigenvalues of the charge operator. Proc. Roy. Soc. London. Ser. A. **230** (1955), 382–389.

Källen [Helv. Phys. Acata **26** (1953), 755–760; MR **15**, 587] has shown that Schwinger's definition [Phys. Rev. (2) **75** (1949), 651–679; MR **10**, 663] of charge renormalization from the requirement that the dielectric constant of the vacuum becomes unity for a light wave also ensures that the expectation value of charge for the one-electron state equals e. In this paper it is shown that with this definition, a state corresponding to n_- incoming electrons, n_+ incoming positrons and no incoming photons is an eigen-state of the total charge operator Q, with the eigenvalue $e(n_--n_+)$. *A. Salam* (Cambridge, England).

Lévy, Maurice M. Effect of renormalization on meson-nucleon S-scattering. Phys. Rev. (2) **98** (1955), 1470–1478.

This paper continues the discussion of an earlier paper [Phys. Rev. (2) **94** (1954), 460–468; MR **16**, 102]. The (second-order) covariant integral equation is renormalized and solved approximately. The S-space shifts are computed for the two isotopic spin values ($\frac{1}{2}$ and $\frac{3}{2}$) and compared with experiment. For $G^2/4\pi=7.5$, a reasonable fit is obtained. *A. Salam* (Cambridge, England).

See also: Kaschluhn, p. 568; Gatto, p. 568.

Thermodynamics, Statistical Mechanics

★Sommerfeld, Arnold. Thermodynamics and statistical mechanics. (Lectures on theoretical physics, vol. V.) Edited by F. Bopp and J. Meixner. Translated by J. Kestin. Academic Press Inc., New York, 1956. xviii+401 pp. $7.00.

Translation of the author's Thermodynamik und Statistik [Wiesbaden, 1953; MR **15**, 920].

Popoff, Kyrille. Sur la thermodynamique des processus irréversibles. Z. Angew. Math. Phys. **6** (1955), 378–386.

This is a continuation of a rather long series of papers by the author dealing with the observation that the phenomenological relations of irreversible thermodynamics are first integrals of a system of second-order differential equations of the type $d^2x_i/dt^2=X_i$, with the X_i linear functions of the x_i. This installment treats processes for which the temperature and pressure remain constant and the results are applied, in particular, to the theory of phase. *G. Newell* (Providence, R.I.).

Bergmann, Peter G., and Lebowitz, Joel L. New approach to nonequilibrium processes. Phys. Rev. (2) **99** (1955), 578–587.

The authors consider the time evolution of a system under the influence of external reservoirs, the actions of which at successive times are uncorrelated, so that the evolution of the system is Markoffian. In case one reservoir only is coupled to the system, the latter is shown to approach the thermal equilibrium belonging to the temperature of the reservoir. If there are two or more reservoirs with different temperatures the system tends to a stationary nonequilibrium distribution. For this special case of irreversible processes the Onsager reciprocity relations are established. *L. Van Hove.*

Fujita, Shigeichi. Quasistationary process. I. On entropy production. Kumamoto J. Sci. Ser. A. **2** (1954), no. 1, 51–63.

This paper contains a mixture of two ideas, one of which is correct, namely the following. In the theory of irreversible processes the 'state' of a system is described by n parameters, for instance the forces $X_1, \cdots, X_n$. This implies the assumption that, with respect to all other quantities, the system is in equilibrium under the constraints $X_1, \cdots, X_n$. In practice this may not always be so; in particular, sudden changes of the X_i may give rise to transient effects. *N. G. van Kampen* (Utrecht).

Noll, Walter. Die Herleitung der Grundgleichungen der Thermomechanik der Kontinua aus der statistischen Mechanik. J. Rational Mech. Anal. **4** (1955), 627–646.

The paper deals with a classical system of interacting particles and is devoted to the derivation of the macroscopic balance equations for mass, momentum and energy. This problem has been treated earlier by Irving and Kirkwood [J. Chem. Phys. **18** (1950), 817–829; MR **12**, 230]. The present treatment is characterized by a more orthodox and perhaps a little more rigorous mathematical presentation. In particular the use of delta-functions is avoided. Another difference is that simple closed expressions are given for the stress tensor and the heat flow, the non-kinetic parts of which were given by Irving and Kirkwood as series expansions. Consideration is also given to external forces which cannot be derived from a potential. *L. Van Hove* (Utrecht).

★Hinčin, A. Ya. Symmetric functions on multidimensional surfaces. Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 541–574. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

The author treats systems consisting of n non-interacting components, for which the energy of the system, E, is the sum of n identical energy functions of the Hamiltonian variables of the components. In the following, all asymptotic relations are to be understood to relate to $n\to\infty$, with $E=na$, where a is a positive constant, fixed throughout, so that probability refers to microcanonical weighting on the surface of constant energy E. The following are the principal results. (a) Let e_1 be the energy of a specified component. The limiting distribution function K of e_1 is found, and it is shown that the density of

distribution of e_1 converges to K'. (b) A sequence of constants is found such that, with probability approaching 1, the maximum component energy will differ from the nth constant by a bounded quantity. (c) If $n\psi(u)$ is the number of components having energy $<u$, it is shown that $n^{\frac{1}{2}}[\psi_n(u)-K(u)]$ is, for each u, asymptotically normally distributed, with mean 0 and variance an explicitly specified function of u. (d) If $e_1 \leqq \cdots \leqq e_n$ are now the particle energies in increasing order, a number u_k is found with the property that $n^{\frac{1}{2}}(e_k-u_k)$, when $k/n \to$ const, is asymptotically normally distributed with mean 0 and a specified variance. (Here u_k depends on n.) (e) If $\beta>0$, and if $\varrho_\beta(F_1, F_2)=\sup_{0<u<\infty} |F_1(u)-F_2(u)|\exp(\beta u)$, a positive α is found for which, if $\beta<\alpha$, $\varrho_\beta(\psi_n, K)\to 0$ stochastically. (f) The Gibbs theorem that the micrononicacal distribution of the system on a surface of constant energy leads to the macrocanonical distribution of energy for individual components is generalized by deriving the same conclusion under a considerably less restrictive hypothesis. The microcanonical density is replaced by its product with $F(\psi_n)$. Here F is a positive function defined on a class of distribution functions including K and the ψ_n, continuous in terms of the distance ϱ_β, for some $\beta<\alpha$, and satisfying a certain boundedness condition. *J. L. Doob* (Urbana, Ill.).

Kaschluhn, F. Zu einem approximativen Variationsprinzip in der Quantenstatistik. Ann. Physik (6) **16** (1955), 304–305.

MacDonald and Richardson [Phys. Rev. (2) **96** (1954), 18–21; MR **16**, 204] obtained the density matrix ϱ of a grand canonical ensemble by minimizing $\varrho(\log \varrho-\beta\mu N+\beta H)$, and using certain approximations. It is here shown that the thermodynamic state variables constructed with this ϱ do not, in general, satisfy the correct thermodynamic relations. The additional terms that appear arise from the temperature dependence of the minimizing wave functions. *N. G. van Kampen* (Utrecht).

Gatto, R. The formalism of second quantization in quantum statistical mechanics. Nuovo Cimento (10) **2** (1955), 592–604.

The sum of states is the trace of the operator $\exp[-(H^{(0)}+H')/kT]$ and hence invariant for change of representation. A perturbation expansion in H' is therefore conveniently written in terms of creation and annihilation operators; the trace of the successive terms is then taken by averaging over a canonical ensemble of the unperturbed system. The relevant formulas are derived and the procedure is demonstrated with an example. (There is a discontinuity in the notation between Eqs. (8) and (9).) *N. G. van Kampen* (Utrecht).

Blatt, J. M., Butler, S. T., and Schafroth, M. R. Statistical mechanics of rotating buckets. Phys. Rev. (2) **100** (1955), 481–495.

One expects that a fluid in a rotating vessel has a moment of inertia $I=M\langle x^2+y^2\rangle_{Av}$, where $\langle\ \rangle_{Av}$ denotes a geometrical average over the cross-section of the vessel. In a preceding paper [Phys. Rev. (2) **100** (1955), 476–480], Blatt and Butler showed that this is not true for an ideal Bose gas. The present paper shows that it is true: (i) for classical gases; and (ii) for quantum gases having a finite 'correlation distance' Λ, i.e., a distance beyond which the momenta of two particles cease to be correlated. For an ideal Bose gas below its transition temperature this condition is violated, Λ being of the order of the dimensions of the vessel. A not wholly convincing argument, involving red giants, is used to show that Λ must be finite as soon as the particles interact. *N. G. van Kampen* (Utrecht).

Peretti, Jean. Méthode de calcul de la densité spectrale des fréquences propres de vibration d'un réseau cristallin. C. R. Acad. Sci. Paris **241** (1955), 461–463.

The elastic frequency distribution function of a crystalline lattice is expressed as the Hilbert transform of a function which is itself simply related by an integral formula to the elastic constants of the lattice. This equation is used to rederive the results of the reviewer [Phys. Rev. (2) **89** (1953), 1189–1193; MR **15**, 88] on the singularities of the frequency distribution. *L. Van Hove* (Utrecht).

Peretti, Jean. Sur le spectre de fréquence d'un réseau cristallin. Méthode de résolution approximative de l'équation intégrale du spectre de fréquence. C. R. Acad. Sci. Paris **241** (1955), 544–546.

The Hilbert transform formula derived in the paper reviewed above for the frequency distribution of a crystal lattice is here used to construct a systematic approximation procedure for the determination of the distribution. *L. Van Hove* (Utrecht).

Montroll, Elliott W., and Potts, Renfrey B. Effect of defects on lattice vibrations. Phys. Rev. (2) **100** (1955), 525–543.

This paper develops the theory of the normal vibrations of a crystal lattice of large extent, when it contains one or more localized defects (abnormal masses, spring constants, or topology of neighbors). Most of the analysis concerns the one-dimensional lattice, but some applications of the methods to a three-dimensional simple cubic lattice are given. The defects modify the propagating modes and sometimes introduce discrete non-propagating modes, i.e., modes whose atomic amplitudes approach zero with increasing distance from the defect, and whose frequencies exceed the maximum possible for propagating modes. The natures and frequencies of all the modes are obtainable from a Green's function $g(n, \omega)$ measuring the amplitude of oscillation of atom n in the steady-state response of the perfect lattice to a harmonic force of frequency ω applied to the atom $n=0$. The calculation of the perturbed modes from $g(n, \omega)$ is reduced to an eigenvalue problem in a space of dimensionality equal to the number of units of the lattice which are modified by the defect. Calculation of $g(n, \omega)$ is made analytically, and sums over both types of modes (e.g., the zero point energy) are evaluated by reducing them to contour integrals.

At large distances r from the defect the atomic amplitude in a non-propagating mode decreases asymptotically as $\exp(-Ar)$ and $r^{-1}\exp(-Ar)$ for the one- and three-dimensional problems, respectively. Explicit calculations are given for the modification of the zero-point energy by a defect, or by two like defects a finite distance apart. In the latter case the energy decreases as the defects come closer together. *C. Herring.*

Kitaĭgorodskiĭ, A. I. Theory of connections among structure amplitudes. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 482–484. (Russian)

See also: Vodička, p. 369; Hil'mi, p. 417.